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Perturbations of gaps in the essential spectra of self-adjoint relations

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ABSTRACT
This paper is concerned with the stability of gaps in the essential spectra of self-adjoint relations under non-negative relatively compact perturbation in Hilbert spaces. A stability result about semi-boundedness of self-adjoint relations under relatively bounded perturbation is given. It is shown that the gaps in the essential spectra of self-adjoint relations are invariant under non-negative relatively compact perturbation. The results obtained in the present paper generalize the corresponding results for linear operators to linear relations.

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1. Introduction

Spectral perturbation problems in linear operators have been studied extensively due to its widely applicability, and many elegant results have been obtained (cf., [1,2]). However, the classical perturbation theory of linear operator is not available in many cases. For example, the minimal and maximal operators corresponding to a linear discrete Hamiltonian system or a linear symmetric difference equation are multi-valued and non-densely defined (cf. [3,4]). So we should apply the perturbation theory of multi-valued linear operators to study the above problems. Further, multi-valued linear operator theory may provide some useful tools for the study of spectral problems of generalized indefinite strings that have a strong physical background (cf., [5,6]) and boundary value problems for differential operators [7]. Due to these reasons, it is necessary and urgent to study some topics about multi-valued linear operators,which are also called linear relations or linear subspaces. They are briefly called relations or subspaces in the present paper.

Linear relations were introduced by von Neumann [8], motivated by the need to consider adjoint operators of non-densely defined linear differential operators. The operational calculus of linear relations was developed by Arens [9]. His works were followed by many scholars, and some basic concepts, fundamental properties, self-adjoint extension, resolvent, spectrum and perturbation for linear relations were studied (cf., [3,4,7,10–28]).

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Now, we shall recall some existing results about perturbations of essential spectra of linear relations. In 1998, Cross introduced a concept of essential spectrum of a linear relation and showed that it is stable under relatively compact perturbation with certain additional conditions (see[16, Theorem VII.3.2]). In 2014, Wilcox gave five distinct concepts of essential spectra of linear relations in Banach spaces, and proved that they are stable under relatively compact perturbation with some additional conditions and under compact perturbation, separately (see [26, Theorems 4.4 and 5.3]). In 2016, shi obtained the invariance of essential spectra of self-adjoint relations under compact and relatively compact perturbation with additional conditions, separately (see [20, Theorems 5.1 and 5.2]). Further, we extended these results to more general perturbations in 2018 (see [28, Theorems 4.1 and 4.2]).

In the present paper, enlightened by the methods used in the study of the stability of essential spectra of self-adjoint operators, we shall continue to deeply study the perturbations of essential spectra of self-adjoint relations. Especially, we are concerned with the behavior of gaps in essential spectrum of a self-adjoint relation under a non-negative relatively compact perturbation. It will be shown that some perturbation results about the essential spectra of linear operators can be naturally extended to linear relations. Here, the essential spectrum of a linear relation is defined as the subset of its spectrum consisting of either accumulation points or isolated eigenvalues of infinite multiplicity (see Definition 2.6). Some of the results obtained in the present paper extend the related existing results about self-adjoint operators to self-adjoint relations (see Remarks 3.9 and 4.4).

The rest of this paper is organized as follows. In Section 2, some notations, basic concepts and fundamental results about linear relations are introduced. In Section 3, the perturbations of semi-bounded self-adjoint relations are studied. It is shown that the semi-boundedness of self-adjoint relations is stable under relatively bounded perturbation. Finally, it is proved that the gaps in essential spectrum of a self-adjoint relation is stable under non-negative relatively compact perturbation in Section 4.

2. Preliminaries

In this section, we shall introduce some basic concepts and fundamental results about linear relations, which will be used in the sequent sections.

Let X be a Hilbert space over the complex field $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$. The product space X^2 is still a Hilbert space with the following induced inner product, still denoted by $\langle \cdot, \cdot \rangle$ without any confusion

$$\langle (x_1, y_1), (x_2, y_2) \rangle := \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \quad (x_1, y_1), \quad (x_2, y_2) \in X^2.$$

Any linear subspace $T \subset X^2$ is called a *linear relation (briefly, relation)* of X^2. The domain $D(T)$, range $R(T)$, and null space $N(T)$ of T are defined by

$$D(T) := \{x \in X : (x, y) \in T \text{ for some } y \in X\},$$
$$R(T) := \{y \in X : (x, y) \in T \text{ for some } x \in X\},$$
$$N(T) := \{x \in X : (x, 0) \in T\},$$

respectively. Further, denote

$$T(x) := \{y \in X : (x, y) \in T\},$$

$$T^{-1} := \{(y,x) : (x,y) \in T\}.$$

By $LR(X)$ denotes the set of all linear relations of X^2.

Obviously, we have $T(0) = \{0\}$ if and only if T can determine a linear operator from $D(T)$ into X whose graph is T. For convenience, a linear operator (i.e. single-valued operator) in X will always be identified with a linear relation in X^2 via its graph.

Let T, $A \in LR(X)$ and $\alpha \in \mathbb{C}$. Define

$$\alpha\, T := \{(x, \alpha\, y) : (x,y) \in T\},$$
$$T + A := \{(x, y+z) : (x,y) \in T,\ (x,z) \in A\},$$
$$T - \alpha I := \{(x, y - \alpha x) : (x,y) \in T\}.$$

The *product* of A and T is defined by

$$AT := \{(x,z) \in X^2 : (x,y) \in T,\ (y,z) \in A \text{ for some } y \in X\}.$$

Note that if A and T are both operators, then AT is also an operator. If $T \cap A = \{(0,0)\}$, denote

$$T\dot{+}A := \{(x_1 + x_2, y_1 + y_2) : (x_1, y_1) \in T,\ (x_2, y_2) \in A\}.$$

Further, if T and A are orthogonal, that is, $\langle (x,y), (u,v) \rangle = 0$ for all $(x,y) \in T$ and $(u,v) \in A$, then denote

$$T \oplus A := T\dot{+}A.$$

Let $X \in LR(X)$. The *adjoint* of T is defined by

$$T^* := \{(f,g) \in X^2 : \langle g, x \rangle = \langle f, y \rangle \text{ for all } (x,y) \in T\}.$$

We say T is *Hermitian* in X^2 if $T \subset T^*$, and *self-adjoint* in X^2 if $T = T^*$.

Lemma 2.1 ([23, Proposition 2.1]): *Let $T, A \in LR(X)$. Then $T = T + A - A$ if and only if $A(0) \subset T(0)$ and $D(T) \subset D(A)$.*

Lemma 2.2 ([27, Lemma 5.8]): *Let $T \in LR(X)$ be self-adjoint. If $A \in LR(X)$ be Hermitian and $D(T) \subset D(A)$, then $A(0) \subset T(0)$.*

Arens gave a decomposition of a closed linear relation T in X^2 [9]:

$$T = T_s \oplus T_\infty$$

where

$$T_\infty := \{(0,y) \in X^2 : (0,y) \in T\},\ T_s := T \ominus T_\infty.$$

It can be easily verified that T is an operator if and only if $T = T_s$. Accordingly, T_s and T_∞ are called the operator and pure multi-valued parts of T, respectively. In addition, they satisfy the following properties [9]:

$$D(T_s) = D(T), \quad R(T_s) \subset T(0)^\perp, \quad T_\infty = \{0\} \times T(0).$$

Let $T \in LR(X)$. By Q_T, or simply Q when there is no ambiguity about the relation T, denote the natural quotient map from X onto $X/\overline{T(0)}$. Clearly, QT is an operator [16]. Further, denote $B_X := \{x \in X : \|x\| \le 1\}$.

Definition 2.3: ([16,23]) Let $T \in LR(X, Y)$. For any given $x \in D(T)$, the *norms* of $T(x)$ and T are defined by

$$\|T(x)\| := \|(QT)(x)\|, \|T\| := \|QT\| = \sup\{\|(QT)(x)\| : \ x \in D(T) \cap B_X\},$$

respectively. If $\|T\| < \infty$, T is said to be *bounded*. Further, T is said to be *compact*, if QT is compact.

Lemma 2.4 ([16, Propositions II.1.4 and II.1.5]): *Let $T, A \in LR(X)$. Then,*

(i) $\|T(x)\| = \mathrm{dist}(y, T(0)) = \mathrm{dist}(0, T(x))$ *for $x \in D(T)$ and $y \in T(x)$;*
(ii) $\|T(x) + A(x)\| \leq \|T(x)\| + \|A(x)\|$ *for $x \in D(T) \cap D(A)$;*
(iii) $\|(\alpha T)(x)\| = |\alpha|\|T(x)\|$ *for $\alpha \in \mathbb{C}$ and $x \in D(T)$.*

Lemma 2.5 ([23, Theorem 2.4]): *Let $T \in LR(X)$ be closed. Then*

$$\|T(x)\| = \|T_s(x)\| \quad \text{for } x \in D(T).$$

The following concepts were introduced in [9,22].

Definition 2.6: Let T be a linear relation in X^2.

(1) The set $\rho(T) := \{\lambda \in \mathbb{C} : \ (T - \lambda I)^{-1}$ is a bounded operator defined on $X\}$ is called the *resolvent set* of T.
(2) The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of T.
(3) For $\lambda \in \mathbb{C}$, if there exists $(x, \lambda x) \in T$ for some $x \neq 0$, then λ is called an *eigenvalue* of T, while x is called an *eigenvector* of T with respect to the eigenvalue λ. Further, the set of all the eigenvalues of T is called the *point spectrum* of T, denoted by $\sigma_p(T)$.
(4) The *essential spectrum* $\sigma_e(T)$ of T is the set of those points of $\sigma(T)$ that are either accumulation points of $\sigma(T)$ or isolated eigenvalues of infinite multiplicity.

Lemma 2.7 ([22, Proposition 2.1 and Theorems 2.1, 2.2 and 3.4]): *Let T be a closed Hermitian relation in X^2. Then $T_s = T \cap (T(0)^{\perp})^2$ is a closed Hermitian operator in $T(0)^{\perp}$, and*

$$\begin{aligned} \sigma(T) = \sigma(T_s), \ \sigma_e(T) = \sigma_e(T_s) \ \sigma_p(T) = \sigma_p(T_s) \\ N(T - \lambda I) = N(T_s - \lambda I), \ \lambda \in \sigma_p(T). \end{aligned} \tag{1}$$

Lemma 2.8 ([29, p. 26]): *If T is a self-adjoint relation in X^2, then T_s and T_∞ are self-adjoint relations in $(T(0)^{\perp})^2$ and $T(0)^2$, respectively.*

In the following, we shall recalled the concept of spectral family of a self-adjoint relation, which was introduced by Coddington and Dijksma in [15].

Let T be a self-adjoint relation in X^2. By Lemma 2.8, T_s is a self-adjoint operator in $T(0)^{\perp}$. Then T_s has the following spectral resolution:

$$T_s = \int t dE_s(t),$$

where $\{E_s(t)\}_{t\in\mathbb{R}}$ is the spectral family of T_s in $T(0)^{\perp}$. The *spectral family* of the relation T is defined by

$$E(t) = E_s(t) \oplus O, \quad t \in \mathbb{R}, \tag{2}$$

where O is the zero operator defined on $T(0)$. Denote

$$E(\mathbf{N})x := \int_{\mathbf{N}} dE(t)x, \ \mathbf{N} \subset \mathbb{R}, \quad x \in X.$$

Lemma 2.9: ([22, Theorem 3.9]) *Let $T \in LR(X)$ be self-adjoint.*

(i) *If $a < b$ and $\dim R(E_s(b-) - E_s(a)) = m$, then $\sigma(T) \cap (a, b)$ consists of only isolated eigenvalues of finite multiplicity, and the sum of the multiplicities of these eigenvalues is equal to m*
(ii) *If $\dim R(E_s(b) - E_s(a)) = \infty$, then $\sigma_e(T) \cap [a, b] \neq \emptyset$.*

Remark 2.10: Note that $R(E(t)) = R(E_s(t))$ for every $t \in \mathbb{R}$ by (2), the conditions about $\{E_s(t)\}_{t\in\mathbb{R}}$ in Lemma 2.9 can be replaced by $\{E(t)\}_{t\in\mathbb{R}}$.

3. Perturbations of the semi-boundedness of self-adjoint relations

In this section, we shall discuss the stability of the semi-bundedness of self-adjoint relations under relatively bounded perturbation. We shall first introduce the concepts of relative boundedness and relative compactness of linear relations.

Let $T \in LR(X)$ and X_T denote the space $(D(T), \|\cdot\|_T)$, where

$$\|x\|_T = \|x\| + \|T(x)\|, \ x \in D(T).$$

Define $G_T : X_T \to X$ by $G_T(x) = x$ for $x \in X_T$. G_T is called the *graph operator* of T.

Definition 3.1: [16, Definition VII.2.1] Let $T, A \in LR(X)$ with $D(T) \subset D(A)$.

(1) The linear relation A is said to be T-bounded if there exist non-negative numbers a and b such that

$$\|A(x)\| \leq a\|x\| + b\|T(x)\|, x \in D(T). \tag{3}$$

If A is T-bounded, then the infimum of all numbers $b \geq 0$ for which (3) holds with a constant $a \geq 0$ is called the T-bound of A.
(2) The linear relation A is said to be T-compact (or relatively compact to T) if AG_T is compact, i.e. $A : X_T \to X$ is compact.

Lemma 3.2 ([27, Theorem 5.2]): *Let $T \in LR(X)$ be self-adjoint and $A \in LR(X)$ be Hermitian with $D(T) \subset D(A)$. If A is T-bounded with T-bound< 1, then $T + A$ is also self-adjoint in X^2.*

Next, we recall the definition of lower bounded Hermitian relations, which was introduced in [14].

Definition 3.3: Let T be an Hermitian relation in X^2.

(1) T is said to be bounded from below (above) if there exists a constant $c \in \mathbb{R}$ such that

$$\langle y, x\rangle \geq c\|x\|^2 (\langle y, x\rangle \leq c\|x\|^2), \quad \forall (x, y) \in T,$$

while such a constant c is called a lower (upper) bound of T.

(2) T is said to be non-negative (non-positive) if 0 is a lower (upper) bound of T.

Lemma 3.4 ([22, Theorem 3.2]): *Let $T \in LR(X)$ be self-adjoint. Then T is bounded from below if and only if its spectrum is bounded from below. Moreover, the greatest lower bound of T is equal to* $\min \sigma(T)$.

Lemma 3.5 ([21, Theorem 3.5]): *Let $T \in LR(X)$ be self-adjoint. Then, for each $z \in \rho(T)$, we have that*

$$\|(T - z)^{-1}\| = [\mathrm{dist}(z, \sigma(T))]^{-1}. \tag{4}$$

Lemma 3.6: *Let $T \in LR(X)$ be closed and Hermitian. Then*

$$\|T(T - z)^{-1}\| = \|T_s(T_s - z)^{-1}\| = \sup_{t\in\sigma(T)} \frac{|t|}{|t - z|}, \quad z \in \rho(T). \tag{5}$$

Proof: Since T is closed, $T(0)$ is a closed subspace in X. Suppose that $z \in \rho(T)$. It follows from (1) that $z \in \rho(T_s)$, i.e. $(T_s - z)^{-1}$ is a bounded operator defined on $T(0)^{\perp}$. For any given $x \in X$, it can be decomposed as $x = x_1 + x_2$ with $x_1 \in T(0)^{\perp}$ and $x_2 \in T(0)$. Then $(T - z)^{-1}(x) = (T_s - z)^{-1}(x_1)$ by [28, Corollary 3.3], which together with Lemma 2.5 implies that

$$\begin{aligned}\|T(T - z)^{-1}(x)\| &= \|T(T_s - z)^{-1}(x_1)\| \\ &= \|T_s(T_s - z)^{-1}(x_1)\| \leq \|T_s(T_s - z)^{-1}\|\|x\|.\end{aligned}$$

Thus, $\|T(T - z)^{-1}\| \leq \|T_s(T_s - z)^{-1}\|$.

On the other hand, for any $x \in T(0)^{\perp}$, we have $(T_s - z)^{-1}(x) = (T - z)^{-1}(x)$. Again by Lemma 2.5, we obtain

$$\begin{aligned}\|T_s(T_s - z)^{-1}(x)\| &= \|T_s(T - z)^{-1}(x)\| \\ &= \|T(T - z)^{-1}(x)\| \leq \|T(T - z)^{-1}\|\|x\|.\end{aligned}$$

Consequently, $\|T_s(T_s - z)^{-1}\| \leq \|T(T - z)^{-1}\|$. Hence we have

$$\|T(T - z)^{-1}\| = \|T_s(T_s - z)^{-1}\|.$$

For the second equality in (5), it directly follows from Lemma 2.7 that T_s is a closed Hermitian operator in $T(0)^{\perp}$. Then $\|T_s(T_s - z)^{-1}\| = \sup_{t\in\sigma(T_s)} \frac{|t|}{|t-z|}$ by (3.17) in [1, p.273].

Therefore, (5) holds and this completes the proof. ■

Theorem 3.7: *Let $T \in LR(X)$ be self-adjoint and bounded from below, and $A \in LR(X)$ be Hermitian with $D(T) \subset D(A)$ and T-bound with T-bound < 1. Then $T + A$ is self-adjoint and bounded from below. If β_T is a lower bound of T and the inequality* (3) *holds with $b < 1$, then*

$$\beta = \beta_T - \max\left\{\frac{a}{b-a},\ a + b|\beta_T|\right\} \tag{6}$$

is a lower bound of $T + A$.

Proof: The self-adjointness of $T + A$ is known by Lemma 3.2. By Lemma 3.4, it suffice to show that $(-\infty, \beta) \subset \rho(T + A)$. Given any $\lambda < \beta$, we have $\lambda \in \rho(T)$ again by Lemma 3.4. It follows from (3), (4) and (5) that

$$\begin{aligned}\|A(T - \lambda)^{-1}\| &\leq a\|(T - \lambda)^{-1}\| + b\|T(T - \lambda)^{-1}\| \\ &= a\frac{1}{\operatorname{dist}(\lambda, \sigma(T))} + b \sup_{t\in\sigma(T)} \frac{|t|}{|t - \lambda|} \\ &\leq a(\beta_T - \beta)^{-1} + b\max\{1, |\beta_T|(\beta_T - \beta)^{-1}\} < 1.\end{aligned}$$

Note that $A(0) \subset T(0)$ by Lemma 2.2. It follows from [27, Theorem 3.1] that $\lambda \in \rho(T + A)$. The proof is complete. ■

Theorem 3.8: *Let $T \in LR(X)$ be self-adjoint and non-negative, and $A \in LR(X)$ be Hermitian with $D(T) \subset D(A)$. If $\|A(x)\| \leq \|T(x)\|$ for all $x \in D(T)$, then*

$$|\langle x, y\rangle| \leq \langle x, z\rangle \text{ for all } (x, y) \in A \text{ and } (x, z) \in T.$$

Proof: Let $S = kA$ for every $k \in (-1, 1)$. Then S is Hermitian and T-bounded with T-bound< 1. Note that T is non-negative, we can take $a = 0, b = |k|$, and $\beta_T = 0$ in Theorem 3.7. Hence $\beta = 0$, which yields that $T + kA$ is a self-adjoint and non-negative linear relation in X^2 for every $k \in (-1, 1)$. Consequently, for every $(x, y) \in A$ and $(x, z) \in T$, we have that

$$\langle x, z + ky\rangle \geq 0 \text{ for any } k \in (-1, 1).$$

By letting $k \to \pm 1$, we can obtain that

$$\langle x, z + y\rangle \geq 0 \text{ and } \langle x, z - y\rangle \geq 0 \text{ for all } (x, y) \in A\ (x, z) \in T.$$

Hence, $|\langle x, y\rangle| \leq \langle x, z\rangle$ for all $(x, y) \in A$ and $(x, z) \in T$. This completes the proof. ■

Remark 3.9: Theorems 3.7 and 3.8 generalizes Theorems 9.1 and 9.3 of [2] for liner operators to linear relations, respectively.

4. Gaps in the essential spectra of linear relations under some perturbations

In this section, we shall consider the behavior of gaps in the essential spectra of linear relations under a non-negative relatively compact perturbation.

Lemma 4.1 ([28, Corollary 4.2]): *Let $T \in LR(X)$ be self-adjoint and $A \in LR(X)$ be Hermitian such that $D(T) \subset D(A)$. Assume that A is T-bounded, $T+A$ is self-adjoint and A is T^n-compact for some $n \geq 0$. Then $\sigma_e(T) = \sigma_e(T+A)$.*

The following result is a generalization of [2, Theorem 7.25] for self-adjoint operators to self-adjoint relations.

Lemma 4.2: *Let $T \in LR(X)$ be self-adjoint and $X = X_1 \oplus X_2 \oplus X_3$ with* $\dim X_3 = m < \infty$*. Assume that the projection P_j onto X_j maps $D(T)$ into itself, i.e. $P_jD(T) \subset D(T)$ for j = 1, 2, 3. If*

$$\langle x, y\rangle \begin{cases} \leq a\|x\|^2 & \text{for } (x,y) \in T \text{ with } x \in P_1D(T), \\ \geq b\|x\|^2 & \text{for } (x,y) \in T \text{ with } x \in P_2D(T), \end{cases}$$

then $(a,b) \cap \sigma(T)$ consists of only isolated eigenvalues; the sum of the multiplicities of these eigenvalues is at most m.

Proof: Suppose that $\dim R(E(b-) - E(a)) = \dim R(E_s(b-) - E_s(a)) \geq m+1$. Then, there exists $x \in R(E(b-) - E(a)) \cap X_3^{\perp}$ such that $x \neq 0$. Thus, x can be decomposed as $x = x_1 + x_2$, where $x_1 = P_1x \in P_1D(T)$ and $x_2 = P_2x \in P_2D(T)$. Let $c = (a+b)/2$. We have

$$\|(T_s - c)(x)\|^2 = \int_a^b (t-c)^2 d\|E_s(t)x\|^2 < (\frac{b-a}{2})^2\|f\|^2. \tag{7}$$

Since P_1 and P_2 map $D(T)$ into itself and T is self-adjoint, we have that $x_1, x_2 \in D(T) \subset T(0)^{\perp}$. Let $(x_1, y_1) \in T$ and $(x_2, y_2) \in T$. Then $(x,y) \in T$ with $y = y_1 + y_2$ and $\langle x, y_1\rangle = \langle y, x_1\rangle$ by the self-adjointness of T. Consequently,

$$\begin{aligned} b\|x_2\|^2 &\leq \langle x_2, y_2\rangle = \langle x_2, y\rangle - \langle x - x_1, y_1\rangle \\ &= \langle x_2, y\rangle - \langle y, x_1\rangle + \langle x_1, y_1\rangle \\ &\leq c(\|x_2\|^2 - \|x_1\|^2) + \langle x_2, y - cx\rangle - \langle y - cx, x_1\rangle + a\|x_1\|^2 \\ &= b\|x_2\|^2 - \frac{b-a}{2}\|x\|^2 + \langle x_2, y - cx\rangle - \langle y - cx, x_1\rangle. \end{aligned} \tag{8}$$

Note that there exists $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ such that $\langle y - cx, x_1\rangle = \alpha\langle x_1, y - cx\rangle$. It follows from (7) that

$$\begin{aligned} &|\langle x_2, y - cx\rangle - \langle y - cx, x_1\rangle| = |\langle x_2 - \alpha x_1, y - cx\rangle| \\ &= |\langle x_2 - \alpha x_1, (T_s - c)(x)\rangle| \leq \|x_2 - \alpha x_1\|\|(T_s - c)(x)\| \\ &= (\|x_2\|^2 + \|x_1\|^2)^{1/2}\|(T_s - c)(x)\| = \|x\|\|(T_s - c)(x)\| < \frac{b-a}{2}\|x\|. \end{aligned}$$

This together with (8) implies that $b\|x_2\|^2 < b\|x_2\|^2$, which is a contradiction. Therefore, $\dim R(E(b-) - E(a)) = \dim R(E_s(b-) - E_s(a)) \le m+1$, and then $(a,b) \cap \sigma(T)$ consists of only isolated eigenvalues, and the sum of the multiplicities of these eigenvalues is at most m by Lemma 2.9. The proof is complete. ∎

Theorem 4.3: *Let $T \in LR(X)$ be self-adjoint such that $\sigma_e(T) \cap (c,d) = \emptyset$. Assume that d is not an accumulation point of those eigenvalues of T that belong to (c,d). Further, let $A \in LR(X)$ be Hermitian and non-negative with $D(T) \subset D(A)$. If A is T^2-compact and T-bounded with T-bound< 1. Then*

(i) $\sigma_e(T+A) \cap (c,d) = \emptyset$
(ii) *d is not an accumulation point of those eigenvalues of $T+A$ that belong to (c,d).*

Proof: (i) It directly follows from Lemma 3.2 that $T+A$ is a self-adjoint linear relation in X^2. By the assumption that A is T^2-compact, we can obtain that $\sigma_e(T) = \sigma_e(T+A)$ by Lemma 4.1. Therefore, the first assertion in Theorem 4.3 holds.

(ii) Without loss of generality, we can assume that $(c,d) = (-1,1)$. Science A is T-bounded with T-bound< 1, there exist $a \ge 0$ and $0 \le b < 1$ such that (3) holds. It follows from Lemma 2.1 that $A(0) \subset T(0)$. Hence, for every $s \in [0,1]$, we have that $T = T + sA - sA$ by Lemma 2.2. By (3) and Lemma 2.4, we can obtain that

$$\begin{aligned}\|A(x)\| &\le a\|x\| + b\|(T+sA-sA)(x)\| \\ &\le a\|x\| + b\|(T+sA)(x)\| + sb\|A(x)\|.\end{aligned}$$

This together with the fact that $sb < 1$ implies that

$$\|A(x)\| \le \frac{a}{1-sb}\|x\| + \frac{b}{1-sb}\|(T+sA)(x)\|. \tag{9}$$

Further

$$\max_{s\in[0,1]} \frac{a}{1-sb} = \frac{a}{1-b} \text{ and } \max_{s\in[0,1]} \frac{b}{1-sb} = \frac{b}{1-b}.$$

Set $\gamma = \max\{\frac{a}{1-b}, \frac{b}{1-b}\} + 1$. Then $\gamma > 0$ and for every $s \in [0,1]$, we can get that

$$\|A(x)\| \le \gamma(\|x\| + \|(T+sA)(x)\|), \qquad x \in D(T). \tag{10}$$

Let $s_0 \in [0,1]$ be chosen such that 1 is not an accumulation point of eigenvalues of $T + s_0A$ belonging to $(-1,1)$. Obviously, this holds for $s_0 = 0$ in any event. One can easily show that $T + s_0A$ is self-adjoint and $\sigma_e(T+s_0A) \cap (-1,1) = \sigma_e(T) \cap (-1,1) = \emptyset$ by Lemmas 3.2 and 4.1. Let E_0 and $E_{0,s}$ denote the spectral family of $T + s_0A$ and its operator part $(T + s_0A)_s$, respectively. Then

$$\dim R(E_0(1-) - E_0(0)) = \dim R(E_{0,s}(1-) - E_{0,s}(0)) < \infty$$

by (ii) of Lemma 2.8 and Remark 2.10.

Let $x \in R(E_0(0)) \cap D(T) = R(E_{0,s}(0)) \cap D(T_s)$. For every $(x, y) \in A$ and $(x, z) \in T$, we have $(x, z + s_0 y) \in T + s_0 A$. Then

$$\langle x, z + s_0 y\rangle = \langle x, (T + s_0 A)_s(x)\rangle = \int_{-\infty}^{0} td\langle E_{0,s}(t)x, x\rangle \le 0. \tag{11}$$

This yields that the linear relation $(T + s_0 A)|_{R(E_0(0))\cap D(T)}$ is non-positive. Consequently, $(I - T - s_0 A)|_{R(E_0(0))\cap D(T)}$ is non-negative. By Lemma 2.5 and [28, Proposition 3.2], we have that

$$\begin{aligned}
&\|(I - T - s_0 A)(x)\|^2 = \|(I - (T + s_0 A)_s)(x)\|^2 \\
&= \|x - (T + s_0 A)_s(x)\|^2 \\
&= \langle x, x\rangle - 2\langle x, (T + s_0 A)_s(x)\rangle + \langle (T + s_0 A)_s(x), (T + s_0 A)_s(x)\rangle \\
&= \|x\|^2 + \|(T + s_0 A)_s(x)\|^2 - 2\int_{-\infty}^{0} td\langle E_{0,s}(t)x, x\rangle \\
&\ge \|x\|^2 + \|(T + s_0 A)_s(x)\|^2.
\end{aligned}$$

This together with (10) implies that

$$\|A(x)\|^2 \le 4\gamma^2(\|x\|^2 + \|(T + s_0 A)_s(x)\|^2) \le 4\gamma^2\|(I - T - s_0 A)(x)\|^2.$$

Hence,

$$\|A(x)\| \le 2\gamma\|(I - T - s_0 A)(x)\|.$$

Note that A and $(I - T - s_0 A)|_{R(E_0(0))\cap D(T)}$ is non-negative. It follows from Theorem 3.8 that

$$\langle x, y\rangle \le 2\gamma\langle x, x - z - s_0 y\rangle \tag{12}$$

for every $(x, y) \in A$ and $(x, z) \in T$ with $x \in R(E_0(0)) \cap D(T)$.

Let $s > s_0$ with $s - s_0 \le 1/(4\gamma)$. By (10), we have

$$\|(s - s_0)A(x)\| \le \frac{1}{4}(\|x\| + \|(T + s_0 A)(x)\|). \tag{13}$$

Note that the relation $T + s_0 A$ is self-adjoint. Hence, $T + sA = (T + s_0 A) + (s - s_0)A$ is a self-adjoint relation in X^2 by Lemma 3.2. For every $(x, y) \in A$ and $(x, z) \in T$ with $x \in R(E_0(0)) \cap D(T)$, by (11) and (12), we can obtain that

$$\begin{aligned}
&\langle x, z + sy\rangle = \langle x, z + s_0 y\rangle + (s - s_0)\langle x, y\rangle \\
&\le \langle x, z + s_0 y\rangle + \frac{1}{4\gamma}\langle x, y\rangle \le \langle x, z + s_0 y\rangle + \frac{1}{2}\langle x, x - z - s_0 y\rangle \\
&= \frac{1}{2}\langle x, z + s_0 y\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x\|^2.
\end{aligned} \tag{14}$$

On the other hand, for every $(x, y) \in A$ and $(x, z) \in T$ with $x \in R(I - E_0(1-)) \cap D(T)$, we have

$$\langle x, z + sy \rangle = \langle x, z + s_0 y \rangle + (s - s_0)\langle x, y \rangle \geq \langle x, z + s_0 y \rangle \tag{15}$$

since A is non-negative. Set $f = z + s_0 y$, it can be decomposed as $f = f_1 + f_2$, where $f_1 \in T(0)^{\perp}$ and $f_2 \in T(0)$. Note that $(T + s_0 A)(0) = T(0)$ by Lemma 2.2. Then $f_1 = (T + s_0 A)_s(x)$, and consequently,

$$\begin{aligned}\langle x, f \rangle = \langle x, f_1 \rangle = \langle x, (T + s_0 A)_s(x) \rangle &= \int_1^{+\infty} t d\langle E_{0,s}(t)x, x \rangle \\ &= \|(I - E_{0,s}(1-))(x)\|^2 = \|x\|^2\end{aligned}$$

by Lemma 2.7. Thus

$$\langle x, z + sy \rangle \geq \|x\|^2.$$

This together with (14) and Lemma 4.2 implies that $(\frac{1}{2}, 1) \cap \sigma(T + sA)$ consists of only isolated eigenvalues, and thus 1 is not an accumulation point of the eigenvalues of $T + sA$ from $(-1, 1)$.

Let $m \in \mathbb{N}$ satisfy $m \geq 4c$, and $\mu = 1/m$. Then $\mu \leq 1/(4c)$. By setting $s_0 = 0$ and $s = \mu$, we can obtain that 1 is not an accumulation point of the eigenvalues of $T + \mu A$ from $(-1, 1)$ by the above proof. Let $s_0 = \mu$ and $s = 2\mu$, one can also get that 1 is not an accumulation point of the eigenvalues of $T + 2\mu A$ from $(-1, 1)$ again by the above proof. We can proceed step by step. Then 1 is not an accumulation point of the eigenvalues of $T + 3\mu A$, $T + 4\mu A, \cdots, T + m\mu A = T + A$ from $(-1, 1)$. This completes the proof. ■

Remark 4.4: Theorem 4.3 extend [2, Theorem 9.14] for linear operators to linear relations.

Remark 4.5: Note that the essential spectrum of a self-adjoint can be only determined by its operator part (Lemma 2.7). So it is natural to take into account perturbations of operator parts of the unperturbed and perturbed linear relations. However, the operator part of a summation of two closed linear relation is not equal to the summation of their operator parts in general (see Example 4.6). In our fourth coming paper, we shall discuss the relationships between perturbations of operator parts of the unperturbed and perturbed linear relations.

Example 4.6: Let $X := l^2 = \{x = \{x_i\}_{i=1}^{\infty} \subset \mathbb{C} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with the inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \bar{y}_i, \quad x = \{x_i\}_{i=1}^{\infty}, \quad y = \{y_i\}_{i=1}^{\infty} \in X.$$

For each $n \geq 1$, let $e_n = \{e_{ni}\}_{i=1}^{\infty} \in X$ with $e_{nn} = 1$ and $e_{ni} = 0$ for all $i \neq n$. Further, let $X_1 = (span\{e_1\})^{\perp} = \{x = \{x_i\}_{i=1}^{\infty} \in X : x_1 = 0\}$.

Define

$$T = \{(x, x + ce_1) : \ x \in X_1, \ c \in \mathbb{C}\},$$

and define the operator A by :

$$A(x)_1 = x_2, \quad A(x)_2 = x_1, \quad A(x)_i = 0, \quad i \geq 3, \quad x = \{x_i\}_{i=1}^{\infty} \in X. \tag{16}$$

Then $D(T) = X_1 \subset D(A) = X$ and $T(0) = span\{e_1\}$. So T is multi-valued. It follows from [28, Example 4.1] that T is a self-adjoint linear relation in X^2.

Now, we show that A is Hermitian. For any $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in X$, we have that

$$\langle x, A(y)\rangle = \langle A(x), y\rangle = \bar{y}_2 x_1 + \bar{y}_1 x_2,$$

which implies that A is Hermitian.

Note that A is bounded. Hence, A is T-bounded with T-bounded zero, and consequently, $T + A$ is self-adjoint in X^2 by Lemma 3.2. On the other hand,

$$T + A = \{(x, x + A(x) + ce_1) : \ x \in X_1, \ c \in \mathbb{C}\},$$

and

$$D(T + A) = D(T) = X_1 \subset D(A), \quad (T + A)(0) = T(0) = span\{e_1\}.$$

Thus, $(T + A)(0)^{\perp} = T(0)^{\perp} = X_1$. By Lemma 2.7, one can get that

$$(T + A)_s = (T + A) \cap ((T + A)(0)^{\perp})^2 = (T + A) \cap X_1^2.$$

Hence, $(x, x + A(x) + ce_1) \in (T + A)_s$ if and only if $x_1 \in X_1$ and $x + A(x) + ce_1 \in X_1$. Note that $x \in D(T) = X_1$, we have that $(x, x + A(x) + ce_1) \in (T + A)_s$ if and only if $A(x) + ce_1 \in X_1$, which equivalent to $A(x)_1 + c = 0$, that is $x_2 + c = 0$ by (16), which implies that $c = -x_2$. Therefore,

$$(T + A)_s(x) = x + A(x) - x_2 e_1, \quad x \in X_1. \tag{17}$$

In addition, it is obviously that

$$T_s(x) = x, \quad x \in X_1. \tag{18}$$

It follows from (16)–(18) that

$$(T + A)_s(x) \neq T_s(x) + A(x), \quad x \in X_1 \text{ with } x_2 \neq 0.$$

Remark 4.7: As we have mentioned in the introduction, it is very important to study spectral properties of multi-valued operators because the minimal and maximal operators corresponding to a linear symmetric difference equation are multi-valued and non-densely defined (cf. [3,4]). The results obtained in the present paper may be available in this case. We shall apply these results to study invariance of essential spectra for second-order symmetric difference equations under some perturbations in our forthcoming paper.

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