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Research Article

Relationships between Perturbations of a Linear Relation and Its Operator Part

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In this study, we establish some relationships between perturbations of a linear relation and its operator part by constructing an operator, which is induced by two linear relations including their closedness, hermiticity, self-adjointness, various spectra, defect indices, and perturbation terms.

1. Introduction

Motivated by the study for the adjoint of nondensely defined linear differential operators, the concept of linear relations, a natural generalization of linear operators, was introduced in [1]. Along with the development of operator theory, the spectral theory for linear relations has been extensively studied and has important applications to several problems (cf., [2–14]). It is worth mentioning that the spectra of linear relations may provide some useful tools for the study of certain operators, such as the maximal and minimal operators corresponding to linear continuous Hamiltonian systems or symmetric linear difference equations [12, 15], and the inverse of certain operators in the study of some Cauchy problems associated with parabolic type equations in Banach spaces [16].

To the best of our knowledge, there are still many important fundamental problems of linear relations that have neither been studied nor completed. In 1961, Arens showed that every closed linear relation T in a Hilbert space can be decomposed as an operator part T_s and a purely multivalued part T_∞ [17]; this decomposition provides a bridge between linear relations and operators. In 1985, Dijksma and de Snoo proved that the operator part T_s of a self-adjoint relation T is also a self-adjoint in the Hilbert space $T(0)^\perp$ [18]. Later, Shi et al. established some relationships between the spectra and various spectra of a closed relation and its operator part as well [19]. Enlightened by these works, the main idea of this study was to construct a linear operator A_T, which is induced by two linear relations T and A such that some perturbations of T can be consistent with its operator part T_s and the various spectra of $T + A$ and $T_s + A_T$ are identical. Consequently, one can study some perturbation problems about linear relations by using these results and related existing results about operators.

The rest of this study is organized as follows. In Section 2, some preliminary and auxiliary results that will be used in the sequel are introduced. In addition, a new linear operator A_T is introduced, which is induced by two linear relations T and A, and its properties are studied. In Section 3, the decomposition of a linear relation is given in terms of reducing subspaces, and relationships between a relation and its decomposition parts are established, including their closedness, hermiticity, self-adjointness, defect indices, and spectra. Using these relationships, the corresponding relationships between relation $T + A$ and operator $T_s + A_T$ are given in case if T is closed. Further, a concept of trace class linear relations is introduced and relationships between various perturbations of a closed relation T and its operator part T_s are discussed. It is shown that if relation A is a relatively bounded or compact perturbation term of T, or A belongs to degenerate or trace class linear relations, then A_T is the same perturbation term of T_s, respectively (see-Theorems 9–12).

2. Preliminaries

In this section, we shall recall some basic concepts, give some fundamental results on linear relations, and introduce a new linear operator induced by two relations and study its properties, which will be used in the sequent sections. This section is divided into three subsections.

2.1. Some Basic Concepts and Fundamental Results of Linear Relations. Let $X, Y, Z, \cdots$ denote normed spaces over a number field $\mathbf{K}$. The norm of $X \times Y$ is defined by

$$\|(x, y)\| = \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2}, x \in X, y \in Y, \tag{1}$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms of the spaces X and Y, respectively, still denoted by $\|\cdot\|$ without any confusion. Similarly, if X and Y are inner product spaces, then the inner product of $X \times Y$ is defined by

$$\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle, (x_1, y_1), (x_2, y_2) \in X \times Y. \tag{2}$$

Obviously, if X and Y are complete, then $X \times Y$ is also complete.

$\mathbf{C}$ and $\mathbf{R}$ denote the sets of complex and real numbers, respectively.

Any linear subspace $T \subset X \times Y$ is called a *linear relation (briefly, relation or subspace)* of $X \times Y$. $LR(X, Y)$ denotes the set of all the linear relations of $X \times Y$. In the case where $X = Y$, $LR(X)$ denotes $LR(X, Y)$, briefly.

The domain $D(T)$, range $R(T)$, and null space $N(T)$ of T are, respectively, defined by

$$\begin{aligned} D(T) &= \{x \in X\colon (x, y) \in T \text{ for some } y \in Y\},\\ R(T) &= \{y \in Y\colon (x, y) \in T \text{ for some } x \in X\},\\ N(T) &= \{x \in X\colon (x, 0) \in T\}. \end{aligned} \tag{3}$$

Furthermore, it denotes the following:

$$\begin{aligned} T(x) &= \{y \in Y\colon (x, y) \in T\}\\ T^{-1} &= \{(y, x)\colon (x, y) \in T\}. \end{aligned} \tag{4}$$

T is said to be *injective* if $N(T) = 0$, and *surjective* if $R(T) = Y$.

It is evident that $T(0) = \{0\}$ if and only if T can uniquely determine a linear operator from $D(T)$ into X whose graph is T. For convenience, a linear operator (i.e., single-valued operator) from X to Y will always be identified with a relation in $X \times Y$ via its graph. In addition, $N(T) = \{0\}$ if and only if $T^{-1}(0) = \{0\}$, i.e., T is injective if and only if T^{-1} is a linear operator. Further, T is said to be closed if $T = \overline{T}$, where $\overline{T}$ is the closure of T.

Let $T, A \in LR(X, Y)$ and $\alpha \in \mathbf{K}$. We define

$$\begin{aligned} \alpha T &= \{(x, \alpha y)(x, y) \in T\},\\ T + A &= \{(x, y + z)(x, y) \in T, (x, z) \in A\},\\ T - \alpha&\{(x, y - \alpha x)(x, y) \in T\} \text{in the case that } Y = X. \end{aligned} \tag{5}$$

If $T \cap A = \{(0, 0)\}$, we denote

$$T \dot{+} A = \{(x_1 + x_2, y_1 + y_2)(x_1, y_1) \in T, (x_2, y_2) \in A\}. \tag{6}$$

Further, in the case that X and Y are inner product spaces, if T and A are orthogonal, that is, $\langle (x, y), (u, v)\rangle = 0$ for all $(x, y) \in T$ and $(u, v) \in A$, then it denotes the following:

$$T \oplus A = T \dot{+} A. \tag{7}$$

The *product* of linear relations $T \in LR(X, Y)$ and $A \in LR(Y, Z)$ is defined by (see [17])

$$AT = \{(x, z) \in X \times Z (x, y) \in T, (y, z) \in A \text{ for some } y \in Y\}. \tag{8}$$

Note that if A and T are operators, then AT is also an operator.

In the following, we shall briefly recall the concepts of bounded and compact relations, which were introduced in [20, 21].

Let X and Y be normed spaces and $T \in LR(X, Y)$. By Q_T, or simply Q when there is no ambiguity about the relation T, we denote the natural quotient map from Y onto $Y/\overline{T(0)}$. Clearly, QT is an operator [20]. Furthermore, it denotes that $B_X\colon = \{x \in X\colon \|x\| \leq 1\}$.

For any given $x \in D(T)$, the *norms* of $T(x)$ and T are defined by

$$\begin{aligned} \|T(x)\| &= \|(QT)(x)\|,\\ \|T\| &= \|QT\| = \sup\{\|(QT)(x)\| x \in D(T) \cap B_X\}, \end{aligned} \tag{9}$$

respectively. If $\|T\| < \infty$, then T is said to be *bounded.*

T is said to be *compact* if QT is compact.

It is evident that T is compact if and only if for every bounded sequence $\{x_n\}_{n=1}^{\infty} \subset D(T)$ and $\{(QT)(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in $Y/\overline{T(0)}$. Moreover, if T is compact, then T is bounded [20] (Corollary V.2.3).

The following results come from [20], Proposition II.1.4, and [22], Corollary 1.

Lemma 1. *Let X and Y be normed spaces and $T \in LR(X, Y)$. Then,*

(i) $\|T(x)\| = d(y, T(0)) = d(0, T(x))$ *for every* $x \in D(T)$ *and* $y \in T(x)$;

(ii) $\|T\| = \sup\{\|T(x)\|: x \in D(T) \cap B_X\}$;

(iii) *For any given* $\{x_n\}_{n=1}^{\infty} \subset D(T)$, $(Q_T T)(x_n) \longrightarrow [y]$ *as* $n \longrightarrow \infty$ *for some* $[y] \in Y/\overline{T(0)}$ *if and only if for each* $n \geq 1$, *there exists* $y_n \in T(x_n)$ *such that* $y_n \longrightarrow y$ *as* $n \longrightarrow \infty$.

Definition 1 (see [19, 23, 24]). Let X be a Banach space over the complex field $\mathbf{C}$ and $T \in LR(X)$.

(1) The set $\rho(T): = \{\lambda \in \mathbf{C}: (T - \lambda)^{-1}$ is a bounde d linear operator de fined on $X\}$ is called the *resolvent set* of T.

(2) The set $\sigma(T): = \mathbf{C}/\rho(T)$ is called the *spectrum* of T.

(3) For $\lambda \in \mathbf{C}$, if there exists $(x, \lambda x) \in T$ for some $x \neq 0$, then λ is called an *eigenvalue* of T, while x is called an *eigenvector* of T with respect to the eigenvalue λ. $N(T - \lambda)$ is called the *eigensubspace* of λ, and dim $N(T - \lambda)$ is called the *multiplicity* of λ. Further, the set of all the eigenvalues of T is called the *point spectrum* of T, denoted by $\sigma_p(T)$.

(4) The *essential spectrum*$\sigma_e(T)$ of T is the set of those points of $\sigma(T)$ that are either accumulation points of $\sigma(T)$ or isolated eigenvalues of infinite multiplicity.

(5) The set $\sigma_d(T): = \sigma(T)/\sigma_e(T)$ is called the *discrete spectrum* of T.

2.2. Concepts and Fundamental Properties of Self-Adjoint Linear Relations. In this subsection, we shall briefly recall the concept of self-adjoint relations and another classification of the spectrum of a self-adjoint relation by its spectral family and some fundamental properties of them.

In this part, X is always assumed to be a complex Hilbert space.

Let $X \in LR(X)$. The adjoint of T is defined by

$$T^* = \{(f, g) \in X^2 \langle g, x\rangle = \langle f, y\rangle \text{for all} (x, y) \in T\}. \tag{10}$$

T is said to be *Hermitian* in X^2 if $T \subset T^*$ and said to be *self-adjoint* in X^2 if $T = T^*$.

Arens introduced the following decomposition for a closed linear relation T in X^2 [17]:

$$T = T_s \oplus T_\infty, \tag{11}$$

where

$$T_\infty = \{(0, y) \in X^2: (0, y) \in T\}, T_s = T \ominus T_\infty. \tag{12}$$

It can be easily verified that T_s is an operator, and T is an operator if and only if $T = T_s$. T_s and T_∞ are called the operator and pure multivalued parts of T, respectively. In addition, they satisfy the following properties:

$$\begin{gathered} D(T_s) = D(T), \overline{D(T_s)} = \overline{D(T)} = T^*(0)^\perp, \\ R(T_s) \subset T(0)^\perp, T_\infty = \{0\} \times T(0). \end{gathered} \tag{13}$$

If T is an Hermitian, it is evident that

$$D(T) \subset T^*(0)^\perp \subset T(0)^\perp. \tag{14}$$

Throughout the present study, the resolvent set and spectrum of T_s mean those of T_s are restricted to $(T(0)^\perp)^2$.

Lemma 2 (see [Proposition 2.1, Theorems 2.1, 2.2 and 3.4 in [19]). *Let T be a closed Hermitian relation in X^2. Then,*

$$\begin{aligned} T_s &= T \cap (T(0)^\perp)^2, \\ T_\infty &= T \cap T(0)^2. \end{aligned} \tag{15}$$

T_s is a closed Hermitian operator in $T(0)^\perp$, and

$$\begin{gathered} \rho(T) = \rho(T_s), \sigma(T) = \sigma(T_s), \sigma(T_\infty) = \varnothing, \\ \sigma_p(T) = \sigma_p(T_s), N(T - \lambda) = N(T_s - \lambda), \lambda \in \sigma_p(T). \end{gathered} \tag{16}$$

Further, if T is a self-adjoint relation in X^2, then

$$\sigma_e(T) = \sigma_e(T_s), \sigma_d(T) = \sigma_d(T_s). \tag{17}$$

Remark 1. It follows from Theorems 2.1 *and* 2.2 *in* [19] that (17) *also holds ifTis a closed Hermitian relation in*X^2.

Lemma 3 (see p. 26 in [18]). *If T is a self-adjoint relation in X^2, then T_s and T_∞ are self-adjoint relations in $(T(0)^{\perp 2}$ and $T(0)^2$, respectively.*

Lemma 4 (see Theorem 2.5 in [19]). *Let T be an Hermitian relation in X^2. Then, T is self-adjoint in X^2 if and only if $R(T - \lambda) = R(T - \overline{\lambda}) = X$ for some $\lambda \in \mathbf{C}$.*

Next, we shall briefly recall another classification of the spectrum of a self-adjoint relation by its spectral family, including continuous spectrum, singular continuous spectrum, absolutely continuous spectrum, and singular spectrum and their some properties (see [19]). The concept of spectral family of a self-adjoint relation was introduced by Coddington and Dijksma in [8].

Let T be a self-adjoint relation in X^2. By Lemma 3, T_s is a self-adjoint operator in $T(0)^\perp$. Then, T_s has the following spectral resolution:

$$T_s = \int t dE_s(t), \tag{18}$$

where $\{E_s(t)\}_{t\in\mathbf{R}}$ is the spectral family of T_s in $T(0)^{\perp}$. The *spectral family* of the relation T is defined by

$$E(t) = E_s(t) \oplus O, t \in \mathbf{R}, \tag{19}$$

where O is the zero operator defined on $T(0)$. We denote

$$E(\mathbf{N})f = \int_{\mathbf{N}} dE(t)f, \mathbf{N} \subset \mathbf{R}, f \in X. \tag{20}$$

Let X_P denote the closed linear hull of all the eigenvectors of T and $X_C{:}\ = X_P^{\perp}$. They are called the *discontinuous* and *continuous subspaces* in X with respect to T, respectively. Further, we denote

$$X_{SC} = \{f \in X_C \text{there exists a Borel null set } \mathbf{N} \text{ such that } E(\mathbf{N})f = f\}, \tag{21}$$

$X_{AC}{:}\ = X_C \ominus X_{SC}$, and $X_S{:}\ = X_P \oplus X_{SC}$, which are called the singular continuous, absolutely continuous, and singular subspaces in X with respect to T.

The (spectral) discontinuous, continuous, singular continuous, absolutely continuous, and singular parts of T are defined by

$$\begin{aligned} T_P &= T \cap X_P^2, \\ T_C &= T \cap X_C^2, \\ T_{SC} &= T \cap X_{SC}^2, \\ T_{AC} &= T \cap X_{AC}^2, \\ T_S &= T \cap X_S^2, \end{aligned} \tag{22}$$

respectively.

Definition 2 (see Definition 4.1 in [19]). Let T be a self-adjoint relation in X^2. The spectra of T_C, T_{SC}, T_{AC}, and T_S are called the *continuous spectrum*, *singular continuous spectrum*, *absolutely continuous spectrum*, and *singular spectrum* of T, respectively, denoted by $\sigma_c(T)$, $\sigma_{sc}(T)$, $\sigma_{ac}(T)$, and $\sigma_s(T)$, respectively.

Lemma 5 (see Theorem 4.1 in [19]). *Let T be a self-adjoint relation in X^2. Then,*

$$\begin{aligned} \sigma(T_P) &= \sigma((T_s)_P), \\ \sigma_r(T) &= \sigma_r(T_s), \\ r &= c, ac, sc, s. \end{aligned} \tag{23}$$

2.3. An Operator Induced by Two Linear Relations. In this subsection, we shall first introduce a new linear operator induced by two linear relations, which plays an important role in the present study, and then study its some properties. Further, X is assumed to be a complex Hilbert space.

Let T and A be two linear relations in X^2 with $D(T) \subset D(A)$ and $A(0) \subset T(0)$. P_T denotes the following orthogonal projection:

$$P_T{:}\ X \longrightarrow T(0)^{\perp}. \tag{24}$$

It defines that

$$A_T = P_T A|_{\overline{D(T)}\cap D(A)}. \tag{25}$$

It follows from $A(0) \subset T(0)$ that $A_T(0) = P_T A(0) \subset P_T T(0) = 0$. This means A_T is single-valued. Further, it is evident that $D(T) \subset D(A_T) \subset \overline{D(T)}$ and $R(A_T) \subset T(0)^{\perp}$. Consequently, $\overline{D(A_T)} = \overline{D(T)}$. Note that $A_T = A|_{\overline{D(T)}\cap D(A)}$ in the case that T is single-valued.

Proposition 1. *Let $T, A \in LR(X)$ with $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and $D(T) \subset T(0)^{\perp}$, then*

$$A_T|_{D(T)} = (T + A - T) \cap \left(T(0)^{\perp}\right)^2. \tag{26}$$

Proof. We suppose that T is closed and $D(T) \subset T(0)^{\perp}$. Then, $T(0)$ is closed by Proposition II.5.3 in [20]. Consequently, X can be decomposed as

$$X = T(0)^{\perp} \oplus T(0). \tag{27}$$

Since $A(0) \subset T(0)$, we have that

$$(T + A - T)(x) = A(x) + T(0) = \{y\} + T(0), y \in A(x), x \in D(T). \tag{28}$$

Hence, $(T + A - T) \cap (T(0)^{\perp})^2$ is single-valued.

We first show that $(T + A - T) \cap (T(0)^{\perp})^2 \subset A_T|_{D(T)}$. For any $(x, y) \in (T + A - T) \cap (T(0)^{\perp})^2$, there exist $(x, y_1) \in A$ and $(0, y_2) \in T$ such that $y = y_1 + y_2$ by (28). Let $y_1 = y_{11} + y_{12}$, where $y_{11} \in T(0)^{\perp}$ and $y_{12} \in T(0)$. Then, $y = y_{11} + y_{12} + y_2$. Note that $y, y_{11} \in T(0)^{\perp}$ and $y_{12}, y_2 \in T(0)$. We have $y = y_{11}$. Hence, $A_T(x) = P_T(y_1) = y_{11} = y$. Therefore, $(T + A - T) \cap (T(0)^{\perp})^2 \subset A_T|_{D(T)}$.

Now, we consider the inverse. For any given $(x, y) \in A_T|_{D(T)}$, there exists $z, f \in X$ such that $(x, z) \in A$,

$(z, y) \in P_T$, and $(x, f) \in T$ since $x \in D(T)$. It follows from (11) that (x, f) can be decomposed as $(x, f) = (x, f_1) + (0, f_2)$ with $(x, f_1) \in T_s$ and $(0, f_2) \in T_\infty$. Let $z = z_1 + z_2$, where $z_1 \in T(0)^\perp$ and $z_2 \in T(0)$. Then, $y = z_1$. Since $A(0) \subset T(0)$, we have $T(0) = (T + A)(0)$. Hence, $(0, z_2) \in T + A$. Note that when $(x, f + z) \in T + A$, we get $(x, f + y) \in T + A$. This together with $(x, f) \in T$ implies that $(x, y) \in T + A - T$. Since $x \in D(T) \subset T(0)^\perp$ and $y \in T(0)^\perp$, we have $(x, y) \in (T(0)^\perp)^2$. Consequently, $A_T|_{D(T)} \subset (T + A - T) \cap (T(0)^\perp)^2$.

Therefore, (26) holds. This completes the proof.

The following result is a direct consequence of (14) and Proposition 1. □

Corollary 1. *Let $T, A \in LR(X)$ with $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and Hermitian, then*

$$A_T|_{D(T)} = (T + A - T) \cap \left(T(0)^\perp\right)^2. \tag{29}$$

Now, we give a decomposition of $T + A$ in the case that T is closed. This decomposition plays an important role in the study of some properties about linear relations in the present study.

Theorem 1. *Let $T, A \in LR(X)$ with $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed, then*

$$T + A = (T_s + A_T) \oplus T_\infty. \tag{30}$$

Proof. We suppose that T is closed. It is evident that $T_s + A_T$ and T_∞ are orthogonal.

For any $(x, y) \in T + A$, there exists $y_1, y_2 \in X$ such that $(x, y_1) \in T$, $(x, y_2) \in A$, and $y = y_1 + y_2$. It follows from (11) that $(x, y_1) = (x, y_{11}) + (0, y_{12})$ with $(x, y_{11}) \in T_s$ and $(0, y_{12}) \in T_\infty$. Let $y_2 = y_{21} + y_{22}$, where $y_{21} \in T(0)^\perp$ and $y_{22} \in T(0)$. Then, $(x, y_{21}) \in A_T$ and $(0, y_{12} + y_{22}) \in T_\infty$. Hence, $(x, y) = (x, y_{11} + y_{21}) + (0, y_{12} + y_{22})$ belongs to $(T_s + A_T) \oplus T_\infty$. This implies that $T + A \subset (T_s + A_T) \oplus T_\infty$.

On the other hand, for any given $(x, y) \in (T_s + A_T) \oplus T_\infty$, it can be decomposed as $(x, y) = (x, y_1) + (0, y_2)$ with $(x, y_1) \in T_s + A_T$ and $(0, y_2) \in T_\infty \subset T$. Let $y_1 = y_{11} + y_{12}$, where $(x, y_{11}) \in T_s \subset T$ and $(x, y_{12}) \in A_T$. Given any $z \in A(x)$, it can be decomposed as $z = z_1 + z_2$ with $z_1 \in T(0)^\perp$ and $z_2 \in T(0)$, and we have that $y_{12} = z_1 = z - z_2$ and $(x, y_{11} + y_2 - z_2) \in T$. Consequently,

$$(x, y) = (x, y_1 + y_2) = (x, y_{11} + y_2 - z_2 + z) \in T + A, \tag{31}$$

which yields that $(T_s + A_T) \oplus T_\infty \subset T + A$.

Therefore, (30) holds and the proof is complete. □

Corollary 2. *Let $T, A \in LR(X)$ with $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T and $T + A$ are closed, then*

$$(T + A)_s = T_s + A_T. \tag{32}$$

Proof. Suppose that T and $T + A$ are closed. It follows from $A(0) \subset T(0)$ that $T_\infty = (T + A)_\infty$. Then, $T + A = (T + A)_s \oplus T_\infty$ by (11), which together with (30) implies $(T + A)_s = T_s + A_T$. The proof is complete. □

Remark 2. In Theorem 3.1 in [25], *Shi showed that $(T + A)_s = PT_s + QA_s$ under some conditions, where P: $T(0)^\perp \longrightarrow (T + A)\ \ (0)^\perp$ and Q: $A(0)^\perp \longrightarrow\ \ (T + A)(0)^\perp$ are orthogonal projections. Note that $(T + A)(0) = T(0)$ by the assumption that $A(0) \subset T(0)$. Then, P is an identity mapping from $T(0)^\perp$ onto itself and $QA_s|_{\overline{D(T) \cap D(A)}} = A_T$. Hence,* Theorem 3.1 *in* [25] is consistent with the result in Corollary 2 in the case that A is closed. Further, Theorem 3.2 in [25] can be directly *derived from* Corollary 2.

3. Relationships between Perturbations of T and T_s

In this section, we shall investigate the relationships between properties of $T + A$ and $T_s + A_T$ and the perturbation terms of T and T_s. Note that the relation $T + A$ can be decomposed as (30), we shall consider a general decomposition, which is induced by reducing subspaces and discuss the relationships between a relation and its decomposition parts in Subsection 3.1. Using these results, the corresponding relationships between relation $T + A$ and operator $T_s + A_T$ are given in Subsection 3.2. Further, relationships between various perturbations of closed relation T and its operator part T_s are studied in Subsection 3.3.

3.1. Decomposition of Relations. The following concept of reducing subspace for a linear relation in Banach spaces can be extended to the corresponding one in Hilbert spaces (cf., [18]).

Let X be a Banach space. Suppose that X has the decomposition

$$X = X_1 \dot{+} X_2, \tag{33}$$

where X_1 and X_2 are closed subspaces of X and $X_1 \cap X_2 = \{0\}$. Let P: $X \longrightarrow X_1$ be the projection on X_1 along with X_2 and $T \in LR(X)$. We denote

$$P^{(2)}T = \{(Px, Py)\,(x, y) \in T\}. \tag{34}$$

It is clear that $T \cap X_1^2 \subset P^{(2)}T$.

If $P^{(2)}T \subset T$, then X_1 is called a *reducing subspace* of T. We also say that X_1 reduces T or T is reduced by X_1. In this case, one has that

$$T \cap X_1^2 = P^{(2)}T. \tag{35}$$

It can be easily verified that X_1 reduces T if and only if X_2 reduces T. Further, if X_1 reduces T, then

$$T = T_1 \dot{+} T_2, \tag{36}$$

where

$$T_i = T \cap X_i^2, \quad i = 1, 2. \tag{37}$$

Further, we suppose that X is a Hilbert space and $X_1 \perp X_2$. If T is reduced by X_1, then

$$T = T_1 \oplus T_2, \tag{38}$$

where T_1 and T_2 are defined by (37).

Now, we give a relationship between the closedness of T and its decomposition parts.

Proposition 2. *Let X be a Banach space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i\,(i = 1, 2)$ be defined by (37). Then, T is a closed relation in X^2 if and only if T_i is a closed relation in X_i^2 for each $i = 1, 2$.*

Proof. It is evident that T_i is closed in X_i^2 for $i = 1, 2$ if T is closed.

We suppose that T_i is closed in X_i^2 for $i = 1, 2$. Given any sequence $\{(x_n, y_n)\}_{n=1}^{\infty} \subset T$ with $(x_n, y_n) \longrightarrow (x, y)$ as $n \longrightarrow \infty$, there exist $(x_{n,i}, y_{n,i}) \in T_i, i = 1, 2$ such that $(x_n, y_n) = (x_{n,1}, y_{n,1}) + (x_{n,2}, y_{n,2})$ by (36). Note that X_i is closed for $i = 1, 2$. Then, $P\colon X \longrightarrow X_1$ is bounded by closed graph theorem. So, there exist $x_i, y_i \in X$ such that $(x_{n,i}, y_{n,i}) \longrightarrow (x_i, y_i)$ as $n \longrightarrow \infty$ for $i = 1, 2$. Consequently, $(x, y) = (x_1, y_1) + (x_2, y_2)$ with $(x_i, y_i) \in T_i$ since T_i is closed in X_i^2 for $i = 1, 2$. Hence, $(x, y) \in T$ again by (36). Therefore, T is a closed relation in X^2. This completes the proof.

In the following, we shall discuss the relationships between the hermiticity and self-adjointness of T and its decomposition parts, respectively. □

Proposition 3. *Let X be a Hilbert space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i\,(i = 1, 2)$ be defined by (37) with $X_1 \perp X_2$. Then, T is an Hermitian relation in X^2 if and only if T_i is an Hermitian relation in X_i^2 for each $i = 1, 2$.*

Proof. Obviously, T_i is an Hermitian relation in X_i^2 for each $i = 1, 2$ if T is Hermitian.

We suppose that T_i is an Hermitian relation in X_i^2 for $i = 1, 2$. Given any $(x, y), (f, g) \in T$, there exist $(x_1, y_1), (f_1, g_1) \in T_1$ and $(x_2, y_2), (f_2, g_2) \in T_2$ such that $(x, y) = (x_1, y_1) + (x_2, y_2)$ and $(f, g) = (f_1, g_1) + (f_2, g_2)$ by (38). Since T_i is Hermitian in X_i^2 for $i = 1, 2$, we have that

$$\begin{aligned}\langle f, y\rangle &= \langle f_1 + f_2, y_1 + y_2\rangle = \langle f_1, y_1\rangle + \langle f_2, y_2\rangle \\ &= \langle g_1, x_1\rangle + \langle g_2, x_2\rangle = \langle g_1 + g_2, x_1 + x_2\rangle = \langle g, x\rangle\end{aligned} \tag{39}$$

Therefore, T is an Hermitian. The proof is complete. □

Lemma 6 (see p. 26 in [18]). *Let X be a Hilbert space and $T \in LR(X)$.*

(i) Let X_1 and X_2 be closed subspaces in X and $T_i \subset X_i^2, i = 1, 2$. If $T = T_1 \oplus T_2$ and $X = X_1 \oplus X_2$, then X_i is a reducing subspace of T, $i = 1, 2$.

(ii) If T is a self-adjoint relation in X^2 and X_1 reduces T with $X_1 \perp X_2$, then T_i defined by (37) is a self-adjoint relation in X_i^2 for $i = 1, 2$.

Proposition 4. *Let X be a Hilbert space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i\,(i = 1, 2)$ be defined by (37) with $X_1 \perp X_2$. Then, T is a self-adjoint relation in X^2 if and only if T_i is a self-adjoint relation in X_i^2 for each $i = 1, 2$.*

Proof. The necessity directly follows from (ii) of Lemma 6. Now, we consider the sufficiency. Since T_j is a self-adjoint relation in X_j^2, we have $R(T_j \pm i) = X_j$ for $j = 1, 2$ by Lemma 4. So, it follows from (37) and (38) that

$$R(T \pm i) = R(T_1 \pm i) \oplus R(T_2 \pm i) = X_1 \oplus X_2 = X, \tag{40}$$

which implies that T is a self-adjoint relation in X^2 by Lemma 4 and Proposition 3. This completes the proof.

Now, we discuss the relationships between the defect indices of T and its decomposition parts. □

Definition 3 (see Definition 2.3 in [26]). Let X be a Hilbert space and $T \in LR(X)$. The subspace $R(T - \lambda)^{\perp}$ is called the *defect space* of T and λ, and the number $\beta(T, \lambda)\colon = \dim R(T - \lambda)^{\perp}$ is called the *defect index* of T and λ.

Let X be a Hilbert space. It follows from Theorem 2.3 in [26] that $\beta(T, \lambda)$ is constant in the upper and lower half-planes if T is an Hermitian relation in X^2. In this case, it denotes that

$$d_+(T) = \beta(T, -i), d_-(T) = \beta(T, i), \tag{41}$$

which are called the positive and negative defect indexes of T, respectively. The pair $(d_+(T), d_-(T))$ is called the defect indices of T (see [26]).

Proposition 5. *Let X be a Hilbert space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i\,(i = 1, 2)$ be defined by (37) with $X_1 \perp X_2$. Then,*

$$\beta(T, \lambda) = \beta(T_1, \lambda) + \beta(T_2, \lambda), \lambda \in \mathbf{C}, \tag{42}$$

where T_i is regarded as a relation in X_i^2 for $i = 1, 2$.

Proof. Given any $\lambda \in \mathbf{C}$, it suffices to show that

$$R(T - \lambda)^{\perp} = R(T_1 - \lambda)^{\perp} \oplus R(T_2 - \lambda)^{\perp}, \tag{43}$$

□

where $R(T_i - \lambda)^{\perp}$ is the orthogonal complement of $R(T_i - \lambda)$ in X_i for $i = 1, 2$.

Given any $y \in R(T - \lambda)^{\perp}$, there exist $y_1 \in X_1$ and $y_2 \in X_2$ such that $y = y_1 + y_2$. So, for every $f \in R(T_1 - \lambda) \subset R(T - \lambda) \cap X_1$ and $g \in R(T_1 - \lambda) \subset R(T - \lambda) \cap X_2$, we have $\langle y_1, f\rangle = \langle y, f\rangle = 0$ and $\langle y_2, g\rangle = \langle y, g\rangle = 0$. Hence, $y_i \in R(T_i - \lambda)^{\perp}$ for $i = 1, 2$. And consequently, $R(T - \lambda)^{\perp} \subset R(T_1 - \lambda)^{\perp} \oplus R(T_2 - \lambda)^{\perp}$.

On the other hand, for any $y \in R(T_1 - \lambda)^{\perp} \oplus R(T_2 - \lambda)^{\perp}$, it can be decomposed as $y = y_1 + y_2$ with $y_i \in R(T_i - \lambda)^{\perp} \subset X_i, i = 1, 2$. For any $f \in R(T - \lambda)$, there exists $f_i \in R(T_i - \lambda) \subset X_i$, $i = 1,2$ such that $f = f_1 + f_2$ by (37) and (38). Hence,

$$\langle y, f\rangle = \langle y_1 + y_2, f_1 + f_2\rangle = \langle y_1, f_1\rangle + \langle y_2, f_2\rangle = 0, \tag{44}$$

which implies that $y \in R(T-\lambda)^{\perp}$. Then, $R(T_1-\lambda)^{\perp} \oplus R(T_2-\lambda)^{\perp} \subset R(T-\lambda)^{\perp}$. Therefore, (43) holds, which yields that (42) holds and this completes the proof.

The following result can be directly derived from Propositions 3 and 5.

Corollary 3. *Let X be a Hilbert space and $T \in LR(X)$ be Hermitian and reduced by $X_1 \subset X$. Furthermore, let $T_i (i = 1,2)$ be defined by (37) with $X_1 \perp X_2$. Then,*

$$d_+(T) = d_+(T_1) + d_+(T_2), d_-(T) = d_-(T_1) + d_-(T_2), \tag{45}$$

where T_i is regarded as a relation in X_i^2 for $i = 1,2$.

In the following, we shall investigate the relationships among the spectra and various spectra of T and its decomposition parts, including point spectra, essential spectra, discrete spectra, continuous spectra, singular continuous spectra, absolutely continuous spectra, and singular spectra.

The following result is a generalization of self-adjoint case in Hilbert spaces Proposition 3.2 in [19].

Theorem 2. *Let X be a Banach space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i (i = 1,2)$ be defined by (37). Then,*

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2), \rho(T) = \rho(T_1) \cap \rho(T_2), \tag{46}$$

where $\sigma(T_i)$ and $\rho(T_i)$ are the spectrum and resolvent set of T_i in X_i^2 for $i = 1,2$, respectively.

Proof. It suffices to show that the second relation in (46) holds, which implies that $\sigma(T) = \sigma(T_1) \cup \sigma(T_2)$.

We first show that $\rho(T) \subset \rho(T_1) \cap \rho(T_2)$. For any given $\lambda \in \rho(T)$, $R(T-\lambda) = X$and there exists a constant $c > 0$ such that for all $(x, y) \in T$,

$$\|x\| \le c\|y - \lambda x\|. \tag{47}$$

Obviously, (47) holds for all $(x, y) \in T_1$ since $T_1 \subset T$. Now, we show that $T_1 - \lambda$ is surjective in X_1. For any given $z \in X_1$, there exists $(x, y) \in T$ such that $z = y - \lambda x$ since $R(T-\lambda) = X$. It follows from (36) that (x, y) can be decomposed as $(x, y) = (x_1, y_1) + (x_2, y_2)$ with $(x_i, y_i) \in T_i$ and $i = 1,2$. Then, $z = y_1 - \lambda x_1 + x_2 - \lambda y_2$. Because $z, x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, we have $z = y_1 - \lambda x_1$ and $y_2 - \lambda x_2 = 0$, and thus, $z \in R(T_1 - \lambda)$. Consequently, $T_1 - \lambda$ is surjective in X_1. Hence, $\lambda \in \rho(T_1)$. With a similar argument, one can show $\lambda \in \rho(T_2)$. Then, $\lambda \in \rho(T_1) \cap \rho(T_2)$. Therefore, $\rho(T) \subset \rho(T_1) \cap \rho(T_2)$.

Next, we consider the inverse inclusion. For any given $\lambda \in \rho(T_1) \cap \rho(T_2)$, we have $R(T_i - \lambda) = X_i$ for $i = 1,2$, and there is a constant $b > 0$ such that

$$\|x_i\| \le b\|y_i - \lambda x_i\|, \forall (x_i, y_i) \in T_i, i = 1,2. \tag{48}$$

For any $z \in X$, there exist $z_1 \in X_1$ and $z_2 \in X_2$ such that $z = z_1 + z_2$. Then, there is $(x_i, y_i) \in T_i$ such that $z_i = y_i - \lambda x_i$ for $i = 1,2$, which implies that $z = y_1 + y_2 - \lambda(x_1 + x_2)$. Further, by (36), we can get that $(x_1 + x_2, y_1 + y_2) \in T$. Hence, $z \in R(T-\lambda)$, and consequently, $R(T-\lambda) = X$. Note that $P\colon X \longrightarrow X_1$ is bounded by the closed graph theorem. There exists a constant $M > 0$ such that

$$\|P(x)\| \le M\|x\|, \|(1-P)(x)\| \le M\|x\| \forall x \in X. \tag{49}$$

For any $(x', y') \in T$, it follows from (36) that (x', y') can be uniquely decomposed as $(x', y') = (x_1', y_1') + (x_1', y_2')$ with $(x_i', y_i') \in T_i$ and $i = 1,2$. By utilizing (48) and (49), one can get that

$$\begin{aligned}\|x'\| &\le \|x_1'\| + \|x_2'\| \le b\left(\|y_1' - \lambda x_1'\| + \|y_2' - \lambda x_2'\|\right)\\ &= b\left(\|P(y' - \lambda x')\| + \|(1-P)(y' - \lambda x')\|\right) \le 2bM\|y' - \lambda x'\|,\end{aligned} \tag{50}$$

which yields that $(T-\lambda)^{-1}$ is a bounded linear operator defined on X, and consequently, $\lambda \in \rho(T)$. It follows that $\rho(T_1) \cap \rho(T_2) \subset \rho(T + A)$. Therefore, the second relation in (46) holds. The proof is complete.

Now, we discuss the relationships between the point spectra and essential spectra of T and its decomposition parts. □

Theorem 3. *Let X be a Banach space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i (i = 1,2)$ be defined by (37). Then,*

$$\sigma_p(T) = \sigma_p(T_1) \cup \sigma_p(T_2), N(T-\lambda) = N(T_1 - \lambda) \dotplus N(T_2 - \lambda), \lambda \in \mathbf{C}. \tag{51}$$

Proof. It suffices to show that $N(T-\lambda) = N(T_1 - \lambda) \dotplus N(T_2 - \lambda)$ for every $\lambda \in \mathbf{C}$. It is evident that $N(T_1 - \lambda) \dotplus N(T_2 - \lambda) \subset N(T-\lambda)$ since $T_i \subset T$ and $i = 1,2$. For any $f \in N(T-\lambda)$, we have $(f, \lambda f) \in T$, which can be decomposed as $(f, \lambda f) = (f_1, g_1) + (f_2, g_2)$ by (36), where $(f_i, g_i) \in T_i$ and $i = 1,2$. Note that $\lambda f = g_1 + g_2 = \lambda f_1 + \lambda f_2$ and $f_i, g_i \in X_i$, $i = 1,2$. One can get that $g_i = \lambda f_i$ for $i = 1,2$, which implies that $f_i \in N(T_i - \lambda)$ and $i = 1,2$. This yields that $N(T-\lambda) \subset N(T_1 - \lambda) \dotplus N(T_2 - \lambda)$. Therefore, $N(T-\lambda) = N(T_1 - \lambda) \dotplus N(T_2 - \lambda)$. This completes the proof.

The following result can be easily verified by Theorems 2 and 3. So, its detail proofs are omitted. □

Theorem 4. *Let X be a Banach space, $T \in LR(X)$ be reduced by $X_1 \subset X$, and $T_i (i = 1,2)$ be defined by (47). Then,*

$$\sigma_e(T) = \sigma_e(T_1) \cup \sigma_e(T_2), \sigma_d(T) = \sigma_d(T_1) \cup \sigma_d(T_2). \quad (52)$$

To the end of this subsection, we shall discuss the relationships between the continuous spectra, singular continuous spectra, absolutely continuous spectra, and singular spectra of T and its decomposition parts, separately.

Suppose that X is a Hilbert space and $X_1 \perp X_2$. If $T \in LR(X)$ is self-adjoint and can be decomposed as (38), then T_i is self-adjoint in X_i^2 for $i = 1, 2$ by Lemma 6. Let $T_i = T_{i,s} \oplus T_{i,\infty}$, where $T_{i,\infty} = \{(0, y) \in X_i^2 : (0, y) \in T_i\}$, $T_{i,s} = T_i \ominus T_{i,\infty}$, and $i = 1, 2$. It is evident that

$$T_s = T_{1,s} \oplus T_{2,s}, E(t) = E_1(t) \oplus E_2(t), t \in \mathbf{R}, \quad (53)$$

where $\{E_i(t)\}_{t \in \mathbf{R}}$ is the spectral family of T_i in X_i^2 for $i = 1, 2$.

It follows from the second relation in (51) that

$$X_P = X_{1,P} \oplus X_{2,P}, \quad (54)$$

and consequently,

$$X_C = X_{1,C} \oplus X_{2,C}, \quad (55)$$

where $X_{i,P}$ and $X_{i,C}$ are the discontinuous and continuous subspaces in X_i with respect to T_i for $i = 1, 2$.

By utilizing (53) and (55), we can get that

$$X_{SC} = X_{1,SC} \oplus X_{2,SC}, \quad (56)$$

which together with (54) and (55) implies that

$$X_{AC} = X_{1,AC} \oplus X_{2,AC}, X_S = X_{1,S} \oplus X_{2,S}, \quad (57)$$

where $X_{i,SC}$, $X_{i,AC}$, and $X_{i,S}$ are the singular continuous, absolutely continuous, and singular subspaces in X_i with respect to T_i for $i = 1, 2$.

It is derived from (37), (38), and (54)–(57) that

$$\begin{aligned}
T_P &= T_{1,P} \oplus T_{2,P}, \\
T_C &= T_{1,C} \oplus T_{2,C}, \\
T_{SC} &= T_{1,SC} \oplus T_{2,SC}, \\
T_{AC} &= T_{1,AC} \oplus T_{2,AC}, \\
T_S &= T_{1,S} \oplus T_{2,S}, \\
T_{i,P} &= T_P \cap X_{i,P}^2, \\
T_{i,C} &= T_C \cap X_{i,C}^2, \\
T_{i,SC} &= T_{SC} \cap X_{i,SC}^2, \\
T_{i,AC} &= T_{AC} \cap X_{i,AC}^2, \\
T_{i,S} &= T_S \cap X_{i,S}^2, i = 1, 2.
\end{aligned} \quad (58)$$

where $T_{i,P}$, $T_{i,C}$, $T_{i,SC}$, $T_{i,AC}$, and $T_{i,S}$ are the (spectral) discontinuous, continuous, singular continuous, absolutely continuous, and singular parts of T_i in X_i^2 for $i = 1, 2$, respectively.

Theorem 5. *Let X be a Hilbert space and $T \in LR(X)$ be self-adjoint and reduced by $X_1 \subset X$. Further, let $T_i\,(i = 1, 2)$ be defined by (37) with $X_1 \perp X_2$. Then,*

$$\begin{aligned}
&\sigma(T_P) = \sigma(T_{1,P}) \cup \sigma(T_{2,P}), \\
&\sigma_r(T) = \sigma_r(T_1) \cup \sigma_r(T_2), r = c, ac, sc, s.
\end{aligned} \quad (59)$$

where T_i is regarded as a relation in X_i^2 for $i = 1, 2$.

Proof. This theorem can be directly derived from (i) of Lemma 6, Theorem 2, and (54)–(58). □

3.2. Relationships between $T + A$ and $T_s + A_T$. In this subsection, we shall study the relationships between the properties of $T + A$ and $T_s + A_T$, including their closedness, hermiticity, self-adjointness, various spectra, and defect indices. Further, X is always assumed to be a complex Hilbert space in this part.

Lemma 7. *Let $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and $D(T) \subset T(0)^{\perp}$, then $T + A$ is reduced by $T(0)$.*

Proof. We suppose that T is closed and $D(T) \subset T(0)^{\perp}$. It follows from the assumption $D(T) \subset D(A)$ and $A(0) \subset T(0)$ that

$$T_{\infty} = T \cap T(0)^2 = (T + A) \cap T(0)^2. \quad (60)$$

Note that (15) holds if $D(T) \subset T(0)^{\perp}$, which together with Proposition 1 and Proposition 2.1 in [21] implies that

$$\begin{aligned}
T_s + A_T &= T_s + A_T|_{D(T)} \\
&= T + A - T + T \cap (T(0)^{\perp})^2 \\
&= (T + A) \cap (T(0)^{\perp})^2.
\end{aligned} \quad (61)$$

Therefore, $T + A$ is reduced by $T(0)$ by Theorem 1 and (i) of Lemma 6. This completes the proof.

The following result comes from [6], and we shall give its proof for completeness. □

Lemma 8. *Let $T \in LR(X)$ be closed. Then, T_{∞} is a self-adjoint relation in $T(0)$.*

Proof. We suppose that T is closed. Obviously, T_{∞} is closed in $T(0)$ for any $(0, f), (0, g) \in T_{\infty}$ and $\langle 0, f \rangle = \langle g, 0 \rangle = 0$. Then, T_{∞} is Hermitian in $T(0)$. Note that $R(T_{\infty} \pm i) = T(0)$. Therefore, T_{∞} is a self-adjoint relation in $T(0)$ by Lemma 4. The proof is complete.

By Propositions 2 - 4, Lemmas 7 and 8, (61) and (62), one can easily get the following results. □

Theorem 6. *Let $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and $D(T) \subset T(0)^{\perp}$, then*

(i) $T + A$ is closed if and only if $T_s + A_T$ is closed in $T(0)^{\perp}$;

(ii) $T + A$ is an Hermitian relation in X^2 if and only if $T_s + A_T$ is an Hermitian operator in $T(0)^{\perp}$.

(iii) $T + A$ is a self-adjoint relation in X^2 if and only if $T_s + A_T$ is a self-adjoint operator in $T(0)^{\perp}$.

Now, we give a relationship between the spectra and various spectra of $T + A$ and $T_s + A_T$.

Theorem 7. *Let $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$.*

(i) If T is closed and $D(T) \subset T(0)^{\perp}$, then

$$\sigma_e(T + A) = \sigma_e(T_s + A_T), \sigma_d(T + A) = \sigma_d(T_s + A_T). \tag{62}$$

(ii) If T is closed and $T + A$ is self-adjoint, then

$$\begin{aligned} &\sigma((T + A_P)) = \sigma((T_s + A_T)_P), \\ &\sigma_r(T + A) = \sigma_r(T_s + A_T), r = c, ac, sc, s, \end{aligned} \tag{63}$$

where $T_s + A_T$ is regarded as a relation in $(T(0)^{\perp})^2$.

Proof. The first assertion of Theorem 7 can be easily verified by (16), Theorems 1-4, and Lemma 7. Suppose that $T + A$ is self-adjoint. Then, $D(T) = D(T + A) \subset (T + A)(0)^{\perp} = T(0)^{\perp}$. And consequently, $T(0)$ reduces $T + A$ by Lemma 7. It follows from (16), and Theorems 1, 5, and 6 that (63) holds. This completes the proof. □

Remark 3. By Corollary 2*and* Lemmas 3 and 5, (63) *holds.*

The following results can be easily directly derived from (14) and Theorem 7.

Corollary 4. *Let $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and Hermitian, then*

$$\sigma_e(T + A) = \sigma_e(T_s + A_T), \sigma_d(T + A) = \sigma_d(T_s + A_T), \tag{64}$$

where $T_s + A_T$ is regarded as a relation in $(T(0)^{\perp})^2$.

To the end of this subsection, we shall give a relationship between the defect indices of $T + A$ and $T_s + A_T$.

Theorem 8. *Let $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is closed and $T + A$ is Hermitian, then*

$$d_+(T + A) = d_+(T_s + A_T), d_-(T + A) = d_-(T_s + A_T), \tag{65}$$

where $T_s + A_T$ is regarded as a relation in $(T(0)^{\perp})^2$.

Proof. We suppose that T is closed and $T + A$ is Hermitian. Then, $D(T) = D(T + A) \subset (T + A)(0)^{\perp} = T(0)^{\perp}$. And consequently, $T(0)$ reduces $T + A$ by Lemma 7. It is derived from Theorem 1, Corollary 3, and (ii) of Theorem 6 that (65) holds. Thus, the proof is complete. □

3.3. Relationships between Perturbation Terms of T and T_s. In this subsection, we shall discuss the relationships between the perturbation terms of T and T_s if T is closed. Including relatively bounded and relatively compact perturbation terms, finite rank perturbation term, and trace class perturbation term.

We shall first recall the concepts of relatively bounded and compact relations, which were introduced by Cross [20].

Let X and Y be normed spaces, $T \in LR(X, Y)$, and X_T denote the space $(D(T), \|\cdot\|_T)$, where

$$\|x\|_T = \|x\| + \|T(x)\|, x \in D(T). \tag{66}$$

We then define $G_T \in LR(X_T, X)$ by $G_T(x) = x$ for $x \in X_T$. G_T is called the *graph operator* of T.

Definition 4 (see Definition VII.2.1 in [20]). Let X, Y, and Z be normed spaces, $T \in LR(X, Y)$, and $A \in LR(X, Z)$ with $D(T) \subset D(A)$.

(1) The linear relation A is said to be *T-bounded* if there exist nonnegative numbers a and b such that

$$\|A(x)\| \leq a\|x\| + b\|T(x)\|, x \in D(T). \tag{67}$$

If A is *T-bounded*, then the infimum of all numbers $b \geq 0$ for which (67) holds with a constant $a \geq 0$, is called the *T-bound* of A.

(2) The linear relation A is said to be *T-compact* (or *relatively compact to$T$*) if AG_T is compact, i.e., $A\colon X_T \longrightarrow Z$ is compact.

Lemma 9 (see Lemma 2.7 in [22]). *Let X be a Hilbert space and $T \in LR(X)$ can be decomposed as (11). Then, (11) holds and*

$$\|T(x)\| = \|T_s(x)\|, x \in D(T), \|T\| = \|T_s\|. \tag{68}$$

Lemma 10. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. Then,*

$$\|A_T(x)\| \leq \|A(x)\|, x \in \overline{D(T)} \cap D(A), \|A_T\| \leq \|A\|. \tag{69}$$

Proof. It suffice to show that the first relation in (69) holds. Let $x \in \overline{D(T)} \cap D(A)$. It follows from (i) of Lemma 1 that for any $\epsilon > 0$, there exists $y \in A(x)$ such that $\|y\| < \|A(x)\| + \epsilon$. Let $y = y_1 + y_2$, where $y_1 \in T(0)^{\perp}$ and $y_2 \in \overline{T(0)}$. Then, $A_T(x) = y_1$ by (25). Consequently, $\|A_T(x)\| = \|y_1\| \leq \|y\| < \|A(x)\| + \epsilon$. Hence, $\|A_T(x)\| \leq \|A(x)\|$ by the arbitrariness of ϵ. Therefore, (69) holds and this completes the proof.

By Lemmas 9 and 10, one can easily get the following result. □

Theorem 9. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that T is closed, $D(T) \subset D(A)$, and $A(0) \subset T(0)$. If A is T-bounded with T-bound less that b, then A_T is T_s-bounded with T_s-bound less than b, $b \geq 0$.*

Now, we give a relationship between the relatively compact perturbation of T and T_s.

Theorem 10. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that T is closed, $D(T) \subset D(A)$, and $A(0) \subset T(0)$. If A is T-compact, then A_T is T_s-compact.*

Proof. It follows from (13) and Lemma 9 that $X_T = X_{T_s}$. We suppose that A is T-compact. Then, for any given bounded sequence$\{x_n\}_{n=1}^{\infty}$ in X_T, and $\{(Q_A A)(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence $\{(Q_A A)(x_{n_k})\}_{k=1}^{\infty}$ in $X/\overline{A(0)}$. So, for each $k \geq 1$, there is $y_{n_k} \in A(x_{n_k})$ such that $\{y_{n_k}\}_{k=1}^{\infty}$ is convergent in X by (iii) of Lemma 1. Let $y_{n_k} = y_{n_{k,1}} + y_{n_{k,2}}$ with $y_{n_{k,1}} \in T(0)^{\perp}$ and $y_{n_{k,2}} \in T(0)$. Hence, $A_T(x_{n_k}) = y_{n_{k,1}}$, and $\{y_{n_{k,1}}\}_{k=1}^{\infty}$ is convergent in $T(0)^{\perp}$. This means $\{A_T(x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in $T(0)^{\perp}$. Therefore, A_T is T_s-compact. Thus, the proof is complete.

With a similar argument to that used in the proof of Theorem 10, one can easily show the following results hold. □

Corollary 5. *Let X and Y be Hilbert spaces and $A \in LR(X, Y)$. Then, A is a compact relation if and only if $P_A A$ is a compact operator, where $P_A\colon Y \longrightarrow A(0)^{\perp}$ is an orthogonal projection.*

Corollary 6. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If A is compact, then A_T is compact.*

Proposition 6. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that $D(T) \subset D(A) \cap T(0)^{\perp}$ and $A(0) \subset T(0)$. Then, $A|_{\overline{D(T)} \cap D(A)}$ is an Hermitian relation in X^2 if and only if A_T is an Hermitian operator in $T(0)^{\perp}$.*

Proof. It follows from the assumption $D(T) \subset T(0)^{\perp}$ that $D(A_T) = \overline{D(T)} \cap D(A) \subset T(0)^{\perp}$, which means that A_T is a linear operator in the Hilbert space $T(0)^{\perp}$.

Suppose that $A|_{\overline{D(T)} \cap D(A)}$ is an Hermitian relation in X^2. For any (f, g) and $(h, k) \in A_T$, there exists (f, y), $(h, z) \in A$ such that $y = g + y'$ and $z = k + z'$, where $y', z' \in \overline{T(0)}$. Note that $f, h \in D(A_T) \subset T(0)^{\perp}$ and the assumption that $A|_{\overline{D(T)} \cap D(A)}$ is Hermitian, we get

$$\langle g, h \rangle = \langle y, h \rangle = \langle f, z \rangle = \langle f, k \rangle. \tag{70}$$

Hence, A_T is an Hermitian operator in $T(0)^{\perp}$.

Now, we consider the inverse. We assume that A_T is an Hermitian operator in $T(0)^{\perp}$. We fixed (x, y), $(u, v) \in A|_{\overline{D(T)} \cap D(A)}$. There exist $y_1, v_1 \in T(0)^{\perp}$ and $y_2, v_2 \in \overline{T(0)}$ such that $y = y_1 + y_2$ and $v = v_1 + v_2$. Hence, (x, y_1), $(u, v_1) \in A_T$. By the assumption that A_T is Hermitian and the fact that $x, u \in D(A_T) \subset T(0)^{\perp}$, we get

$$\langle y, u \rangle = \langle y_1, u \rangle = \langle x, v_1 \rangle = \langle x, v \rangle. \tag{71}$$

Therefore, $A|_{\overline{D(T)} \cap D(A)}$ is an Hermitian relation in X^2. This completes the proof.

Let X be a Hilbert space and $T \in LR(X)$. If T is self-adjoint, then $\overline{D(T)} = T^*(0)^{\perp} = T(0)^{\perp}$ by Proposition III.1.4 in [20], which together with Proposition 6, can easily achieve the following result. □

Corollary 7. *Let X be an Hilbert space and $T, A \in LR(X)$, satisfying $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If T is self-adjoint, then $A|_{\overline{D(T)} \cap D(A)}$ is an Hermitian relation in X^2 if and only if A_T is a symmetric operator in $T(0)^{\perp}$, that is, A_T is a densely defined Hermitian operator in $T(0)^{\perp}$.*

Now, we introduce the concept of degenerate linear relation, which is a generation of single-valued case.

Definition 5. Let$X$and$Y$be normed spaces and$A \in LR(X, Y)$.$A$is said to be degenerate if$A$is bounded and$\dim R(A) < \infty$.

Let $A \in LR(X, Y)$ be degenerate. It is evident that $Q_A A$ is degenerate. Hence, $Q_A A$ is compact (see p.160 in [27]), that is, A is compact.

Motivated by the definition of the norm of a linear relation, we introduce the concept of trace class linear relations.

Definition 6. Let$X$and$Y$be Hilbert spaces and$A \in LR(X, Y)$with$D(A) = X$. We say that$A$belongs to trace class relations if$P_A A$belongs to trace class operators, where$P_A\colon Y \longrightarrow A(0)^{\perp}$is the orthogonal projection.

Theorem 11. *Let X be a Hilbert space and $T, A \in LR(X)$ satisfy that $D(T) \subset D(A)$ and $A(0) \subset T(0)$. If A is degenerate, then A_T is also degenerate.*

Proof. We suppose that A is degenerate, that is, $\dim R(A) = m < \infty$ and A is bounded. Then, A_T is bounded by Lemma 10. Let $y_1, y_2, \ldots, y_m$ be a base of $R(A)$. They can be decomposed as $y_i = y_{i,1} + y_{i,2}$ with $y_{i,1} \in T(0)^{\perp}$ and $y_{i,2} \in \overline{T(0)}$, $i = 1, 2, \ldots, m$. Now, we show that for any given $z \in R(A_T) \subset T(0)^{\perp}$, it can be expressed as a linear combination of $y_{i,1}$, $1 \leq i \leq m$. We then set $(x, y) \in A$ with $y = z + y'$, where $y' \in \overline{T(0)}$. There exist c_i and $i = 1, 2, \ldots, m$ such that

$$y = \sum_{i=1}^{m} c_i y_i = \sum_{i=1}^{m} c_i y_{i,1} + \sum_{i=1}^{m} c_i y_{i,2} = z + y'. \tag{72}$$

Note that $z, \sum_{i=1}^{m} c_i y_{i,1} \in T(0)^{\perp}$ and $y', \sum_{i=1}^{m} c_i y_{i,2} \in \overline{T(0)}$, we can get that $z = \sum_{i=1}^{m} c_i y_{i,1}$, which yields that $\dim R(A_T) \leq m < \infty$. Therefore, A_T is degenerate and the proof is complete.

Now, we recall a necessary and sufficient condition about trace operators (cf., Theorem 7.12 in [28]). □

Lemma 11. *Let T be an operator from Hilbert space X into Hilbert space Y with $D(T) = X$. Then, T belongs to trace class operators if and only if there exist sequences $\{f_n\}_{n=1}^{\infty}$ from X and $\{g_n\}_{n=1}^{\infty}$ from Y such that $\|f_n\| = \|g_n\| = 1$ for each $n \geq 1$, and there is a constant sequence $\{c_n\}_{n=1}^{\infty}$ for which $\sum_{n=1}^{\infty} |c_n| < \infty$ and T can be expressed as*

$$T(x) = \sum_{n=1}^{\infty} c_n \langle x, f_n \rangle g_n, x \in X. \tag{73}$$

Remark 4. In the case that$X = Y$in Lemma 11, *Tbelongs to trace class operators if and only if$T$can be expressed as*

$$T(x) = \sum_{n=1}^{\infty} c_n \langle x, f_n \rangle f_n, x \in X, \quad (74)$$

where $\{f_n\}_{n=1}^{\infty}$ form an orthonormal family of eigenvectors of T and the c_n are the associated (repeated) eigenvalues with $\sum_{n=1}^{\infty} |c_n| < \infty$ (cf., [27], p.543).

Theorem 12. *Let X be a Hilbert space and $T, A \in LR(X)$ with $D(A) = X$ and $\overline{D(T)} = T(0)^{\perp}$, and $A(0) \subset T(0)$. If A belongs to trace class relations, then A_T belongs to trace class operators in $T(0)^{\perp}$.*

Proof. The assumptions $D(A) = X$ and $\overline{D(T)} = T(0)^{\perp}$ implies that $D(A_T) = \overline{D(T)} \cap D(A) = T(0)^{\perp}$.

We suppose that A belongs to trace class relations, that is, $P_A A$ belongs to trace class operators. It follows from Lemma 11 that there exist sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ from X and $\{c_n\}_{n=1}^{\infty}$ from $\mathbf{C}$ satisfying $\|f_n\| = \|g_n\| = 1, n \geq 1$, and $\sum_{n=1}^{\infty} |c_n| < \infty$ such that

$$(P_A A)(x) = \sum_{n=1}^{\infty} c_n \langle x, f_n \rangle g_n, x \in X. \quad (75)$$

Let $f_n = f_{n,1} + f_{n,2}$ and $g_n = g_{n,1} + g_{n,2}$ with $f_{n,1}, g_{n,1} \in T(0)^{\perp}$ and $f_{n,2}, g_{n,2} \in \overline{T(0)}$ for $n \geq 1$. We shall show that $A_T = S|_{T(0)^{\perp}}$, where $S = \sum_{n=1}^{\infty} c_n \langle \cdot, f_{n,1} \rangle g_{n,1}$. Given any $x \in D(A_T)$, there is $y \in X$ such that $(x, y) \in A$. Since $A(0) \subset T(0)$, y can be decomposed as $y = y_1 + y_2 + y_3$ with $y_1 \in T(0)^{\perp}$, $y_2 \in A(0)^{\perp} \cap \overline{T(0)}$, and $y_3 \in \overline{A(0)}$. Then, $A_T(x) = y_1$ and $(P_A A)(x) = y_1 + y_2$. By (75) and the fact that $x \in D(A_T) = T(0)^{\perp}$, one can get that

$$\begin{aligned} y_1 + y_2 &= \sum_{n=1}^{\infty} c_n \langle x, f_{n,1} + f_{n,2} \rangle \left(g_{n,1} + g_{n,2}\right) \\ &= \sum_{n=1}^{\infty} c_n \langle x, f_{n,1} \rangle g_{n,1} + \sum_{n=1}^{\infty} c_n \langle x, f_{n,1} \rangle g_{n,2}. \end{aligned} \quad (76)$$

Note that $y_1, \sum_{n=1}^{\infty} c_n \langle \cdot, f_{n,1} \rangle g_{n,1} \in T(0)^{\perp}$ and $y_2, \sum_{n=1}^{\infty} c_n \langle x, f_{n,1} \rangle g_{n,2} \in \overline{T(0)}$, we have that $A_T(x) = y_1 = \sum_{n=1}^{\infty} c_n \langle \cdot, f_{n,1} \rangle g_{n,1}$. Consequently, $A_T \subset S$. Hence, $A_T = S|_{T(0)^{\perp}}$. Let $L: = \left\{n \geq 1: f_{n,1} \neq 0 \text{ and } g_{n,1} \neq 0\right\}$. By setting $f_{n,1}' = f_{n,1}/\|f_{n,1}\|$ and $g_{n,1}' = g_{n,1}/\|g_{n,1}\|$ for each $n \in L$, we can get

$$A_T \backslash (x) = \sum_{n \in L} c_n |f_{n,1}| |g_{n,1}| \langle x, f_{n,1}' \rangle g_{n,1}', x \in T(0)^{\perp}, \quad (77)$$

where $\sum_{n \in L} |c_n| \|f_{n,1}\| \|g_{n,1}\| \leq \sum_{n=1}^{\infty} |c_n| < \infty$. Therefore, A_T belongs to trace class in $T(0)^{\perp}$ by Lemma 11. This completes the whole proof. □

Remark 5. In the present study, we construct a linear operator, which is induced by two linear relations, and then establish the relationships between the perturbation terms of a closed relation and the perturbation terms of its operator part (see Theorems 9–12), and give the relationships between spectrum of a perturbed relation and spectrum of a perturbed operator (see Theorem 7). By using the results obtained in the present study, we shall deeply study stabilities of the spectra of linear relations under some perturbations in our forthcoming study, especially the invariance of the absolutely continuous spectrum of a self-adjoint linear relation under trace class perturbation.

Remark 6. Note that the constructing technique in the present study can also be applied in our previous works. For example, Theorem 5.2 in [29] can be followed by Lemma 3, Corollary 7, Theorems 6 and 9, ([29], Lemma 5.8), and ([27], Theorem V.4.3); Theorem 5.3 in [29] can be followed by Lemmas 2, 3, and 10, Corollaries 4 and 7, Theorem 10, ([29], Lemma 5.8); and ([27], Theorem V.4.10); Theorem 4.1 in [22] can be followed by Lemmas 2 and 3, Corollaries 4 and 7, Theorems 6 and 10, ([29], Lemma 5.8), and ([28], Theorem 9.9), respectively.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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