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*Research Article*

# Relatively Bounded Perturbations of a Closed Linear Relation and Its Adjoint

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This paper is concerned with the relatively bounded perturbations of a closed linear relation and its adjoint in Hilbert spaces. A stability result about orthogonal projections onto the ranges of linear relations is obtained. By using this result, two perturbation theorems for a closed relation and its adjoint are given. These results generalize the corresponding results for single-valued linear operators to linear relations and some of which weaken certain assumptions of the related existing results.

## 1. Introduction

Perturbation theory is one of the main topics in both pure and applied mathematics. In particular, the perturbations of linear operators (i.e., single-valued operators) have received lots of attention and many useful results have been obtained (cf. [1–4]).

However, when considering the adjoint of a nondensely defined linear operator and the minimal and maximal operators corresponding to a linear discrete Hamiltonian system or a linear symmetric difference equation (cf. [5, 6]), the classical perturbation theory of linear operator is not available in these cases. So, we should apply the perturbation theory of multivalued linear operators to study the above problems. Further, multivalued linear operator theory may provide some useful tools for the study of some Cauchy problems associated with parabolic type equations in Banach spaces [7] and boundary value problems for differential operators [8]. Due to these reasons, it is necessary and urgent to study some topics about multivalued linear operators, which are a necessary foundation of research on those related problems about differential or difference operators.

Note that the graph $G(T)$ of a linear operator or multivalued linear operator $T$ from a linear space $X$ to a linear space $Y$ is a linear subspace in the product space $X \times Y$. Further, it is more convenient to introduce concepts of the inverse, closure, and adjoint for linear subspaces. So, we shall directly study linear subspace (briefly, subspace) in the product space $X \times Y$. A subspace is also called a linear relation (briefly, relation). A linear operator always means a single-valued linear operator for convenience in the present paper.

To the best of our knowledge, linear relations were introduced by von Neumann [9], motivated by the need to consider adjoint operators of nondensely defined linear differential operators. The operational calculus of linear relations was developed by Arens [10]. His works were followed by many scholars, and some basic concepts, fundamental properties, self-adjoint extension, resolvent, spectrum, and perturbation for linear relations were studied (cf. [5–8, 11–26]).

There are still many important fundamental problems about linear relations that have neither been studied nor completed. It is shown that the closedness and self-adjointness of linear relation are stable under relatively bounded perturbation (cf. [22, 25]). However, they have not been specifically and thoroughly studied. In the present paper, enlightened by the methods used in [4], we shall deeply study the stability of a closed linear relation and its adjoint under more general relatively bounded perturbation in Hilbert spaces. The spaces $X$ and $Y$ are always assumed to be Hilbert spaces throughout the present paper. The results obtained in the present paper not only cover the related existing results about operators but also some of them weaken the conditions of the corresponding existing results (see Remarks 1 and 2).

The rest of this paper is organized as follows. In Section 2, some notations, basic concepts, and fundamental results about linear relations are introduced. In Section 3, we first give in Theorem 1 a stability result about orthogonal projections onto the ranges of linear relations, which generalize the corresponding result ([4], Theorem 5.25) for linear operators to linear relations. Then, we investigate the relatively bounded perturbations of a closed linear relation and its adjoint. It is shown that the adjoint of the sum of two linear relations is equal to the sum of each adjoint (see Theorem 3).

## 2. Preliminaries

In this section, we shall introduce some basic concepts and give some fundamental results about linear relations, which will be used in the sequent sections.

Let $X$ and $Y$ be Hilbert spaces over the complex field $C$. The norm of $X\times Y$ is defined by

$$\|(x,y)\| = \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2}, \quad x\in X,\ y\in Y, \tag{1}$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms of the spaces $X$ and $Y$, respectively, still denoted by $\|\cdot\|$ without any confusion. The inner product of $X\times Y$ is defined by

$$\begin{aligned}\langle (x_1,y_1),(x_2,y_2)\rangle &= \langle x_1,x_2\rangle + \langle y_1,y_2\rangle,\\ &\qquad (x_1,y_1),(x_2,y_2)\in X\times Y.\end{aligned} \tag{2}$$

Let $E\subset X$, $E^{\perp}$ denotes the orthogonal complement of $E$.

Any linear subspace $T\subset X\times Y$ is called a *linear relation (briefly, relation or subspace)* of $X\times Y$.$LR(X,Y)$ denotes the set of all linear relations of $X\times Y$. In the case that $X=Y$, $LR(X)$ denotes $LR(X,X)$ briefly.

The domain $D(T)$, range $R(T)$, and null space $N(T)$ of $T$ are, respectively, defined by

$$\begin{aligned}D(T) &:= \{x\in X:\ (x,y)\in T \text{ for some } y\in Y\},\\ R(T) &:= \{y\in Y:\ (x,y)\in T \text{ for some } x\in X\},\\ N(T) &:= \{x\in X:\ (x,0)\in T\}.\end{aligned} \tag{3}$$

A linear relation $T$ is said to be injective if $N(T)=0$ and surjective if $R(T)=Y$. Further, denote

$$\begin{aligned}T(x) &:= \{y\in Y:\ (x,y)\in T\},\\ T^{-1} &:= \{(y,x):\ (x,y)\in T\}.\end{aligned} \tag{4}$$

It is evident that $T(0)=\{0\}$ if and only if $T$ can uniquely determine a linear operator from $D(T)$ into $Y$ whose graph is $T$. For convenience, a linear operator (i.e., single-valued operator) from $X$ to $Y$ will always be identified with a relation in $X\times Y$ via its graph. In addition, $N(T)=\{0\}$ if and only if $T^{-1}(0)=\{0\}$, i.e., $T$ is injective if and only if $T^{-1}$ is a linear operator. Further, $T$ is said to be *closed* if $T=\overline{T}$, where $\overline{T}$ is the closure of $T$.

Let $T, A\in LR(X,Y)$ and $\alpha\in C$. Define

$$\begin{aligned}\alpha T &:= \{(x,\alpha y):\ (x,y)\in T\},\\ T+A &:= \{(x,y+z):\ (x,y)\in T,\ (x,z)\in S\},\\ T-\alpha &:= \{(x,y-\alpha x):\ (x,y)\in T\} \text{ in the case that } Y=X.\end{aligned} \tag{5}$$

If $T\cap A=\{(0,0)\}$, denote

$$T\dotplus A := \{(x_1+x_2, y_1+y_2):\ (x_1,y_1)\in T,\ (x_2,y_2)\in A\}. \tag{6}$$

Further, if $T$ and $A$ are orthogonal, that is, $\langle (x,y),(u,v)\rangle = 0$ for all $(x,y)\in T$ and $(u,v)\in A$, then denote

$$T\oplus A := T\dotplus A. \tag{7}$$

The *product* of linear relations $T\in LR(X,Y)$ and $A\in LR(Y,Z)$ is defined as follows (see [10]):

$$\begin{aligned}AT := \{(x,z)\in X\times Z:\ &(x,y)\in T,\\ &(y,z)\in A \text{ for some } y\in Y\}.\end{aligned} \tag{8}$$

Note that if $A$ and $T$ are operators, then $AT$ is also an operator.

*Definition 1.* Let $T\in LR(X,Y)$. The *adjoint* $T^*$ of $T$ is defined as a relation from $Y$ to $X$ by

$$T^* := \{(f,g)\in Y\times X:\ \langle g,x\rangle = \langle f,y\rangle \text{ for all } (x,y)\in T\}. \tag{9}$$

$T$ is said to be *Hermitian* in $X^2$ if $T\subset T^*$ and said to be *self-adjoint* in $X^2$ if $T=T^*$

Let $U$ be the flip-flop operator from $X\times Y$ to $Y\times X$ defined by

$$U(x,y) = (y,-x),\ (x,y)\in X\times Y. \tag{10}$$

It is clear from Definition 1 that

$$T^* = (UT) = UT, \tag{11}$$

where the orthogonal complements refer to the component wise inner product in $Y\times X$ and $X\times Y$, respectively.

Next, we shall briefly recall the concepts of bounded and relatively bounded relations, which were introduced in [17, 22].

Let $T\in LR(X,Y)$. By $Q_T$, or simply $Q$, when there is no ambiguity about the relation $T$, denote the natural quotient map from $Y$ onto $Y/\overline{T(0)}$. Clearly, $QT$ is an operator [17]. Further, denote $B_X := \{x\in X:\ \|x\|\le 1\}$.

*Definition 2.* Let $T\in LR(X,Y)$. For any given $x\in D(T)$, the *norms* of $T(x)$ and $T$ are defined by

$$\begin{aligned}\|T(x)\| &:= \|(QT)(x)\|,\ \|T\| := \|QT\|\\ &= \sup\{\|(QT)(x)\|:\ x\in D(T)\cap B_X\},\end{aligned} \tag{12}$$

respectively. If $\|T\| \langle \infty$, $T$ is said to be *bounded*.

*Definition 3.* Let $T$ and $A$ be two linear relations in $X \times Y$ with $D(T) \subset D(A)$. The linear relation $A$ is said to be $T$ *-bounded* if there exist nonnegative numbers $a$ and $b$ such that

$$\|A(x)\| \le a\|x\| + b\|T(x)\|, \quad x \in D(T). \tag{13}$$

If $A$ is $T$*-bounded*, then the infimum of all numbers $b \ge 0$ for which (13) holds with some constant $a \ge 0$ is called the $T$*-bound* of $A$ .

**Lemma 1** (Propositions II.1.4 and II.1.5 in [17]). *Let $T$ and $S$ be two linear relations in $X \times Y$. Then,*

*(i) $\|T(x)\| = d(y, T(0)) = d(0, T(x))$ for every $x \in D(T)$ and $y \in T(x)$*

*(ii) $\|T(x) + S(x)\| \le \|T(x)\| + \|S(x)\|$ for every $x \in D(T + S)$*

*(iii) $\|\alpha T(x)\| = |\alpha|\|T(x)\|$ for $\alpha \in C$ and $x \in D(T)$*

*Note that the norm $\|T(x)\|$ is not a real norm since the following inequality may not hold in general (see Exercise II.1.12 in [17]):*

$$\|(S - T)(x)\| \ge \|S(x)\| - \|T(x)\|, \quad x \in D(S) \cap D(T). \tag{14}$$

*However, it holds under some conditions.*

**Lemma 2** (Theorem 2.3 in [22]). *Let $T, S \in LR(X, Y)$ satisfy that $D(S) \subset D(T)$ and $\overline{T(0)} \subset \overline{S(0)}$. Then,*

$$\|(S - T)(x)\| \ge \|S(x)\| - \|T(x)\|, \quad x \in D(S). \tag{15}$$

## 3. Main Results

In this section, we shall investigate the relatively bounded perturbations of a closed linear relation and its adjoint. For this purpose, we need to discuss the stability of orthogonal projections onto the ranges of linear relations.

We first give the following auxiliary results about linear operators.

**Lemma 3** (Theorem 4.3 in [4]). *Let $T$ be an operator from $X$ to $Y$ and $M$ be a subspace of $Y$ satisfying that $R(T) \subset \overline{M}$. Then,*

$$\|T(x)\| = \sup\{|\langle Tx, y\rangle| \colon y \in M \text{ with} \|y\| = 1\}, \quad x \in D(T). \tag{16}$$

**Lemma 4** (Theorem 4.33 in [4]). *Let $P_1$ and $P_2$ be two orthogonal projections acting on $X$. Then, we have*

$$\|P_1 - P_2\| = \max\{\rho_{12}\, \rho_{21}\}, \tag{17}$$

*where*

$$\rho_{j,k} = \sup\left\{\left\|P_j h\right\| \colon h \in R(P_k), \|h\| \le 1\right\}. \tag{18}$$

**Theorem 1.** *Let $A, B \in LR(X, Y)$ satisfying $D(A) \subset D(B)$ and $B(0) \subset A(0)$. Assume that there exists a constant $c \ge 0$ such that*

$$\|B(x)\| \le c\|A(x)\| \text{for } x \in D(A). \tag{19}$$

For every $k \in C$, let $P_k$ denote the orthogonal projection onto $\overline{R(A + kB)}$. Then, $\|P_k - P_0\| \longrightarrow 0$ as $k \longrightarrow 0$.

*Proof.* Suppose that $D(A) \subset D(B)$ and $B(0) \subset A(0)$. It follows from (Proposition 2.1 in [22]) that

$$A = A + kB - kB, \tag{20}$$

for every $k \in C$. □

*Case 1.* ($c\rangle 0$). Let $0\langle |k| \langle (1/2c)$. For any $x \in D(A)$, it follows from (19), (20), and Lemma 1 that

$$\begin{aligned} \|B(x)\| &\le c\|(A + kB - kB)(x)\| \\ &\le c\|(A + kB)(x)\| + c|k|\|B(x)\| \\ &\le c\|(A + kB)(x)\| + \left(\frac{1}{2}\right)\|B(x)\|. \end{aligned} \tag{21}$$

Thus,

$$\|B(x)\| \le 2c\|(A + kB)(x)\|. \tag{22}$$

Let $Y = \overline{R(A + kB)} \oplus \overline{R(A + kB)}$. Given any $h \in R(P_0) = \overline{R(A)}$. It can be decomposed as $h = h_1 + h_2$, where $h_1 \in \overline{R(A + kB)}$ and $h_2 \in \overline{R(A + kB)}$. Let $(x, g) \in A + kB$ with $\|g\| = 1$. There exist $g_1, g_2 \in Y$ such that $(x, g_1) \in A$, $(x, g_2) \in B$, and $g = g_1 + kg_2$. By (i) of Lemma 1, we have that for any $\epsilon\rangle 0$, there is $w \in B(x)$ such that

$$\|w\| \langle \|B(x)\| + \epsilon. \tag{23}$$

This together with Lemma 1, (22), and the fact that $g \in R(A + kB)$ implies that

$$\begin{aligned} \|w\| &< 2c\|(A + kB)(x)\| + \epsilon \\ &\le 2c\|g\| + \epsilon = 2c + \epsilon. \end{aligned} \tag{24}$$

Since $(x, w), (x, g_2) \in B$. We have $(0, g_2 - w) \in B$. Hence, $(0, g_2 - w) \in A$ by the assumption that $B(0) \subset A(0)$. Consequently, $(0, k(g_2 - w)) \in A$. Further, noting that $(x, g_1) \in A$, we can get that $(x, g_1 + k(g_2 - w)) \in A$. Then,

$$\begin{aligned} \langle P_k h, g\rangle &= \langle h_1, g\rangle = \langle h, g\rangle = \langle h, g_1 + kg_2\rangle \\ &= \langle h, g_1 + k(g_2 - w) + kw\rangle = \langle h, kw\rangle = \overline{k}\langle h, w\rangle. \end{aligned} \tag{25}$$

It follows from (24) that

$$\left|\langle P_k h, g\rangle\right| \le |k|\|h\|\|w\| \langle\ (2c + \epsilon)|k|\|h\|. \tag{26}$$

Therefore,

$$\left\|P_k h\right\| \le 2c|k|\|h\|, \tag{27}$$

by the arbitrariness of $\epsilon$ and Lemma 3.

On the other hand, for any $h' \in R(P_k) = R(A + kB)$, we can prove in a completely analogous way that

$$\left\|P_0 h'\right\| \le c|k|\left\|h'\right\|, \tag{28}$$

which together with (27) and Lemma 4 yields that $\|P_k - P_0\| \le 2c|k|$. Therefore, $\|P_k - P\|_0 \longrightarrow 0$ as $k \longrightarrow 0$.

*Case 2.* $(c = 0)$. In this case, condition (19) turns into $\|B(x)\| = 0$. Then, $\|w\| \langle \epsilon$ by (23). This together with (25) implies that $|\langle P_k h, g\rangle| \le |k|\|h\|\|w\| \langle \epsilon|k|\|h\|$. Since $\epsilon$ is arbitrary, we have that $\|P_k h\| = 0$. Similarly, we can obtain that $\|P_0 h'\| = 0$ for every $h' \in R(A + kB)$. Therefore, $\|P_k - P_0\| = 0$. This completes the proof.

*Remark 1.* Theorem 1 is a generalization of Theorem 5.25 in [24] for operators to linear relations.

Next, we shall discuss the perturbations of a closed linear relation and its adjoint. We first recall a stability result of the closedness for linear relations.

**Lemma 5** (Theorem 6.3 in [22]). *Let $T, S \in LR(X, Y)$ satisfy that $D(T) \subset D(S)$ and $S(0) \subset T(0)$. If $S$ is $T$-bounded with $T$-bound less that 1, then $T + S$ is closed if and only if $T$ is closed.*

**Theorem 2.** *Let $T, S \in LR(X, Y)$ satisfy that $D(T) \subset D(S)$ and $S(0) \subset T(0)$. Assume that $T$ is closed and $S$ is $T$-bounded with $T$-bound $b$. Further, if $b\rangle 0$, set*

$$\Omega = \left\{|z|\langle \frac{1}{b}\colon T + zS \text{ is closed}\right\}. \tag{29}$$

If $b = 0$, set

$$\Omega = \{|z| \in C\colon T + zS \text{ is closed}\}. \tag{30}$$

Then,

(i) The set $\Omega$ is open.

(ii) For every $z \in \Omega$, let $Q_z$ denote the orthogonal projection from $X \times Y$ onto $T + zS$. Then, $Q_z$ is continuous on $\Omega$ (with respect to the norm topology of $B(X \times Y)$, where $B(X \times Y)$ denote the bounded operators on $X \times Y$).

*Proof*

(i) Suppose that $T$ is closed and $S$ is $T$-bounded with $T$-bound $b$. Set

$$\begin{aligned}\Phi := \{r \ge 0\colon &\text{ there exists } a \ge 0 \text{ such that} \|S(x)\| \le a\|x\| \\ &+ r\|T(x)\|,\ x \in D(T)\}.\end{aligned} \tag{31}$$

Then, $b = \inf\{r\colon r \in \Phi\}$. Note that for every $z \in C$, we have $D(T + zS) = D(T) \subset D(S)$ and $S(0) \subset (T + zS)(0) = T(0)$. Then, by Lemma 2, one has that

$$\|(T + zS)(x)\| \ge \|T(x)\| - \|(zS)(x)\| = \|T(x)\| - |z|\|S(x)\|. \tag{32}$$

This implies that

$$-|z|\|S(x)\| \le \|(T + zS)(x)\| - \|T(x)\|. \tag{33}$$

Let $b\rangle 0$. Given any $z_0 \in \Omega$. There is $\epsilon > 0$ such that $|z_0|\langle 1/(b + \epsilon)$. By the definition of infimum, there exists $r_1 \in \Phi$ such that $r_1 \langle b + \epsilon \langle 1/|z_0|$. Thus, $r_1|z_0|\langle 1$. Further, there is a constant $a_1 \ge 0$ such that $\|S(x)\| \le a_1\|x\| + r_1\|T(x)\|$, which together with (33) yields that

$$\left(1 - r_1\left|z_0\right|\right)\|S(x)\| \le a_1\|x\| + r_1\left\|(T + z_0 S)(x)\right\|. \tag{34}$$

If $b = 0$, for every $z_0 \in \Omega$, there exist $r_2 \in \Phi$ and $a_2 \ge 0$ such that $r_2 \langle 1/|z_0|$ and $\|S(x)\| \le a_2\|x\| + r_2\|T(x)\|$. Again, by (33), we get that

$$\left(1 - r_2\left|z_0\right|\right)\|S(x)\| \le a_2\|x\| + r_2\left\|(T + z_0 S)(x)\right\|, \tag{35}$$

which together with (34) yields that $S$ is $T + z_0 S$-bounded. Hence, $T + zS$ is closed for $z$ sufficiently near to $z_0$ by Lemma 5. Therefore, $\Omega$ is open.

(ii) Define the linear relations $A, B \in LR(X, X \times Y)$ by

$$\begin{aligned} A(x) &= \{(x, f + z_0 g)\colon (x, f) \in T,\ (x, g) \in S\}, \\ B(x) &= \{(0, h)\colon (x, h) \in S\}, \end{aligned} \tag{36}$$

for every $x \in D(T)$. Obviously, $D(A) = D(B) = D(T)$, $A(0) = \{0\} \times T(0)$, and $B(0) = \{0\} \times S(0)$. Thus, $B(0) \subset A(0)$. Further, it is easy to verify that

$$\begin{aligned} \|A(x)\| &= \left\|(T + z_0 S)(x)\right\| + \|x\|, \\ \|B(x)\| &= \|S(x)\|, \end{aligned} \tag{37}$$

and

$$\begin{aligned} R(A) = T + z_0 S, R&(A + kB) \\ &= T + (z_0 + k)S. \end{aligned} \tag{38}$$

Note that $S$ is $T + z_0 S$-bounded. Then, there is a constant $c \ge 0$ such that $\|B(x)\| \le c\|A(x)\|$. Therefore, $Q_z$ is continuous on $\Omega$ by Theorem 1. The proof is complete.

In the following, we shall give a general perturbation result about a closed linear relation and its adjoint. □

**Lemma 6** (Theorem 4.30 in [4]). *Let $M$ and $N$ be closed subspace of $X$, and let $P_M$ and $P_N$ be the orthogonal projections onto $M$ and $N$, respectively. Then, $M \perp N$ if and only if $P_M P_N = 0$ (or $P_N P_M = 0$) and if and only if $P_M + P_N$ is an orthogonal projection.*

**Theorem 3.** *Let $T, S \in LR(X, Y)$ satisfy that $D(T) \subset D(S)$ and $S(0) \subset T(0)$. Assume that $T$ is closed, $S$ is $T$-bounded* *with $T$-bound $b$, and $S^*$ is $T^*$-bounded with $T^*$-bound $b'$. Let $r = max\{b, b'\}$, if $r \rangle 0$, set*

$$\Omega = \left\{ |z| \langle \frac{1}{r} \colon T + zS, T^* + z^* S^* \text{ is closed} \right\}. \tag{39}$$

*If $r = 0$, set*

$$\Omega = \{ |z| \in C \colon T + zS, T^* + z^* S^* \text{ is closed} \}. \tag{40}$$

Further, denote by $\Omega_0$ the connected component of $\Omega$ that contains zero. Then,

$$(T + zS)^* = T^* + z^* S^*, \tag{41}$$

for all $z \in \Omega_0$.

*Proof.* Let $Q_z$ and $Q_z'$ be the orthogonal projections (in $X \times Y$) onto $T + zS$ and $U^{-1}(T^* + z^* S^*)$, respectively, where $U$ is the flip-flop operator defined as (10). It follows from Theorem 2 that the operators $Q_z$ and $Q_z'$ depend continuously on $z \in \Omega$. By (11), we get that $T^\perp = U^{-1} T^*$. Hence, $X \times Y = T \oplus U^{-1} T^*$ and consequently,

$$Q_0 + Q_0' = I. \tag{42}$$

By Proposition III.1.5 in [9], we have that $T^* + z^* S^* \subset (T + zS)^*$, which together with (11) implies that

$$U^{-1}(T^* + z^* S^*) \subset U^{-1}(T + zS)^* = T + zS,\ z \in \Omega. \tag{43}$$

Thus, $Q_z Q_z' = Q_z' Q_z = 0$ for $z \in \Omega$ by Lemma 6. Again by Lemma 6, one has that $I - Q_z - Q_z'$ is an orthogonal projection for any $z \in \Omega$. Consequently, the value of $\|I - Q_z - Q_z'\|$ can be only 0 or 1. Since $\|I - Q_z - Q_z'\|$ depends continuously on $z \in \Omega$, which together with (42) yields that

$$\begin{aligned} \|I - Q_z - Q_z'\| &= \|I - Q_0 - Q_0'\| \\ &= 0,\ z \in \Omega_0. \end{aligned} \tag{44}$$

Hence, $X \times Y = (T + zS) \oplus U^{-1}(T^* + z^* S^*)$. Therefore, $T^* + z^* S^* = U(T + zS) = (T + zS)^*$ for all $z \in \Omega_0$ by (11). This completes the proof.

If the relative bounds of $S$ with respect to $T$ and of $S^*$ with respect to $T^*$ are less than one, then $\{z \in C \colon |z| \le 1\} \subset \Omega_0$ by Lemma 5. Therefore, by Theorem 3, we have the following result. □

**Corollary 1.** *Let $T, S \in LR(X, Y)$ satisfy that $D(T) \subset D(S)$ and $S(0) \subset T(0)$. Assume that $T$ is closed, $S$ is $T$-bounded with $T$-bound less than 1, and $S^*$ is $T^*$-bounded with $T^*$-bound less than 1. Then, $(T + S)^* = T^* + S^*$.*

*Remark 2.* Let $T \in LR(X)$ be self-adjoint and $S \in LR(X)$ be Hermitian with $D(T) \subset D(S)$. Then, $S(0) \subset T(0)$ by Lemma 5.8 in [25]. If $S$ is $T$-bounded with $T$-bound less than one, then $S^*$ is $T^*$-bounded with $T^*$-bound less than one by Corollary III.1.13 in [17], and thus $(T + S)^* = T^* + S^* = T + S$ by Corollary 1, which means $T + S$ is self-adjoint. So, we shall remark that Corollary 1 generalizes Theorem 5.2 in [25] for self-adjoint relations to general linear relations.

*Remark 3.* For studying stabilities of spectra of self-adjoint linear relations, we need to study stabilities of self-adjointness of linear relations. The results obtained in the present paper can be available in this case. Further, we shall apply these results to study stabilities of essential spectra of self-adjoint linear relations under some perturbations in our forthcoming paper.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest.

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