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The structure stability of periodic solutions for first-order uncertain dynamical systems ☆

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Abstract

This paper studies the structural stability of periodic solutions for first-order fuzzy differential equations (FDEs) understood as differential inclusions, i.e., first-order uncertain dynamical systems. The existence and uniqueness of periodic solutions for this first-order fuzzy problems have been obtained on general fuzzy number space. When the forcing function has specific perturbations, the structural stability of the periodic solutions are discussed by using the support function, the Dini Theorem and the Convergence Theorem in the differential inclusion theory.

Keywords: Fuzzy differential equation; Differential inclusion; Periodic solution; Structural stability

1. Introduction

The theory of fuzzy mathematics has its unique advantages in dealing with uncertain factors in the real world. Fuzzy differential equations (FDEs) are often used in mathematical modeling of practical problems with uncertainties. Many studies are about solving fuzzy differential equations. Initial value problems of FDEs [25,32], boundary value problems of FDEs [5,9,12,14,31], periodic problems of FDEs [10,11,13,22–24] have been studied to some extent. And the widely used approaches to deal with the FDEs H-derivatives and Bede's generalized derivatives have gotten abundant achievements (see [2,3,5,7,19,27,30]). But these approaches also have some limitations. For example, the support sets of the fuzzy solutions are nondecreasing in the sense of H-derivatives. So under the H-derivatives, the periodically of FDEs could not be studied very well (see [5,15]). For the simplest periodic problem for first-order fuzzy differential equation: $x' = (-1) \otimes x$, $x(0) = x(1)$ (where $x : [0,\ 1] \to \mathbf{E}_c$, $\mathbf{E}_c$ is continuous fuzzy number space and "$\otimes$" is the operation of product based on Zadeh's Extension Principle) has no solutions under H-derivatives. But

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it can be solved by differential inclusion method (see [11,12]). To overcome this defect, Hüllermeier [20], Diamond et al. [15,17] proposed another approach the differential inclusion method. This method is quite effective in solving the FDEs. Meanwhile it's helpful in the study of the periodically, stability and bifurcation behaviors of FDEs. And differential inclusion method becomes more and more popular (see [1,10,14,15,17,20,21,23,32]).

For first-order FDEs, there are many studies that use different methods to solve them from various angles (see [6,11,23,30,32]). But for the following periodic problems of first-order FDEs:

$$\begin{cases} x'(t) = f(t, x(t)), \\ x(0) = x(T) \end{cases}$$

where $I \overset{\triangle}{=} [0, T]$ $(T > 0)$, $f : I \times \mathbf{E}^n \to \mathbf{E}^n$. By the Theorem 4.1 of [12], there is no nontrivial solution for above FDEs under H-derivatives. In [11], M. Chen et al. consider above FDEs as periodic solutions for first-order fuzzy differential inclusion problems:

$$\begin{cases} \xi'(t) \in f(t, \xi(t)), \\ \xi(0) = \xi(T) \end{cases}$$

where $I \overset{\triangle}{=} [0, T]$ $(T > 0)$, $f : I \times \mathbf{R} \to \mathbf{E}_c$. $\forall \xi \in \mathbf{R}$, $u \in \mathbf{E}_c$, $\xi \in u$ means $u(\xi) = \mu_u(\xi) > 0$, where μ_u is the membership function of u. And the existence and uniqueness of periodic solutions for above problems have been obtained under some conditions on $\mathbf{E}_c$. For general fuzzy number space $\mathbf{E}^n$, the fuzzy numbers on $\mathbf{E}^n$ could not be represented by the new parameter method (see [12]). So in this paper, the metric on $\mathbf{E}^n$ is defined by using the support function. And then the differential inclusion method could be used to discuss the existence and uniqueness of periodic solutions for first-order fuzzy differential inclusion problems on $\mathbf{E}^n$. And the structural stability of periodic solutions has also been discussed and verified by this method, when the forcing function $f(t, \xi(t))$ has some specific perturbations as follows:

$$\begin{cases} \xi'(t) \in f_k(t, \xi(t)), \\ \xi(0) = \xi(T) \end{cases}$$

where $t \in I \overset{\triangle}{=} [0, T] \subset \mathbf{R}$, and $f_k : I \times \mathbf{R}^n \to \mathbf{E}^n$.

This paper is organized as follows. Section 2 provides the basic definitions, properties and theories in the fuzzy space $\mathbf{E}^n$. In Section 3, the existence and uniqueness of periodic solutions have been proved for first-order uncertain dynamical systems under some conditions. In Section 4 we obtained the structural stability of periodic solutions for this problem. In Section 5, the conclusion is given.

2. Preliminaries

In this section, we present the basic concepts, properties and theories of $\mathbf{E}^n$ that are used in this study. Especially, the definition of the metric on $\mathbf{E}^n$ which is defined by using the support function is introduced.

Definition 2.1. [16] Let $\mathbf{D}^n$ be the set of upper semicontinuous normal fuzzy sets with compact supports in $\mathbf{R}^n$ and $\mathbf{E}^n$ be the set of fuzzy convex subsets of $\mathbf{D}^n$.

Theorem 2.1 *(Stacking Theorem). [17] Let $\{A_\alpha \subset \mathbf{R}^n \mid 0 \le \alpha \le 1\}$ be a class of nonempty compact sets satisfying*

(i) $A_\beta \subset A_\alpha$ $(0 \le \alpha \le \beta \le 1)$,

(ii) $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ *for any nondecreasing sequence* $\{\alpha_n\}$ *in* $[0, 1]$ *satisfying* $\alpha_n \to \alpha$.

Then there exists $v \in \mathbf{D}^n$ such that $[v]^\alpha = A_\alpha$ $(0 < \alpha \le 1)$. Especially if A_α is convex, $v \in \mathbf{E}^n$. On the other hand, if $v \in \mathbf{D}^n$, the level set $[v]^\alpha$ satisfies (i) and (ii) above. If $v \in \mathbf{E}^n$, $[v]^\alpha$ is convex.

In above theorem, if A_α belong to some general Banach space, the Stacking Theorem is also holds. And for the fuzzy number space $\mathbf{E}^n$, properties are given as follows.

Definition 2.2. [8] Let $\mathbf{E}^n = \{u | u : \mathbf{R}^n \to [0, 1]$, and u satisfies the following conditions (i)-(v)$\}$,

(i) u is normal, i.e., $\exists m \in \mathbf{R}^n$ such that $u(m) = 1$,
(ii) $[u]^0 = cl\{\xi \in \mathbf{R}^n | u(\xi) > 0\}$ is compact in $\mathbf{R}^n$,
(iii) u is fuzzy convex in $\mathbf{R}^n$,
(iv) u is upper semicontinuous on $\mathbf{R}^n$.

Theorem 2.2. *[8] For $u \in \mathbf{E}^n$, denote $[u]^\alpha = \{\xi \in \mathbf{R}^n | u(\xi) \geq \alpha\}$ $(0 < \alpha \leq 1)$, the following (1)-(3) hold:*

(1) *$[u]^\alpha$ be the nonempty convex compact subsets of $\mathbf{R}^n$, for all $\alpha \in [0, 1]$,*
(2) *$[u]^\beta \subset [u]^\alpha$ $(0 \leq \alpha \leq \beta \leq 1)$,*
(3) *$[u]^\alpha = \bigcap_{n=1}^{\infty} [u]^{\alpha_n}$ for any nondecreasing sequence $\{\alpha_n\}$ in $(0, 1]$ satisfying $\alpha_n \to \alpha$ in $(0, 1]$.*

Conversely, if exists $A^\alpha \subset \mathbf{R}^n$ satisfy (1)-(3) above for all $\alpha \in [0, 1]$, then $\exists u \in \mathbf{E}^n$ such that $[u]^\alpha = A^\alpha$, $\alpha \in (0, 1]$, and $[u]^0 = \bigcup_{\alpha \in (0,1]} [u]^\alpha \subset A^0$.

By above definitions and theorems, the fuzzy number $u \in \mathbf{E}^n$, $(n > 1)$ hasn't the parametric representation which exists in $\mathbf{E}^1$: $u = (u_1, u_2) \in \mathbf{E}^n$. Like in [32], denote $D(u, v) = \sup_{0 \leq \alpha \leq 1} H([u]^\alpha, [v]^\alpha)$, where $u, v \in \mathbf{D}^n$, H is the Hausdorff metric on $P_k(\mathbf{R}^n)$, where $P_k(\mathbf{R}^n)$ be the nonempty compact subset of $\mathbf{R}^n$. And $H(A, B) = \sup\{|\sigma_A(x) - \sigma_B(x)| : x \in S^{n-1}\}$, where $A, B \in P_k(\mathbf{R}^n)$, $S^{n-1} = \{x \in \mathbf{R}^n : ||x||_2 = 1\}$, $|| \cdot ||_2$ be the Euclidean norm on $\mathbf{R}^n$, and $\sigma_A(x) = \sup\{\langle x, y \rangle : y \in A\}$ be the support function of A, $\langle x, y \rangle$ means the inner product of x and y, $x, y \in \mathbf{R}^n$. Then $D(\cdot, \cdot)$ is also the usual Hausdorff metric on $\mathbf{D}^n$ or $\mathbf{E}^n$. And the zero of $\mathbf{E}^n$ is defined by $\hat{0} : \mathbf{R}^n \to [0, 1]$, and

$$\hat{0}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

On the fuzzy space $\mathbf{E}^n$, calculus does also exist. The following properties are used in this paper.

Definition 2.3. [8] Let $f : [a, b] \to \mathbf{E}^n$, $t_0 \in [a, b]$. If $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $D(f(t), f(t_0)) < \varepsilon$ whenever $t \in [a, b]$ and $|t - t_0| < \delta$, then we say that f is continuous at t_0. If f is continuous at each point of $[a, b]$, we say that f is continuous on $[a, b]$.

Definition 2.4. [8] Let $f : [a, b] \to \mathbf{E}^n, t_0 \in [a, b]$. If there exists $f'(t_0) \in \mathbf{E}^n$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $D(\frac{f(t) - f(t_0)}{t - t_0}, f'(t_0)) < \varepsilon$ whenever $t \in [a, b]$, $0 < t - t_0 < \delta$ and $D(\frac{f(t_0) - f(t)}{t_0 - t}, f'(t_0)) < \varepsilon$ whenever $t \in [a, b]$, $0 < t_0 - t < \delta$, then we say that f is derivable at t_0, where $f(t) - f(t_0)$ is the H-difference of $f(t)$ and $f(t_0)$, $f(t_0) - f(t)$ is the H-difference of $f(t_0)$ and $f(t)$. If f is derivable at each point of $[a, b]$, we say that f is derivable on $[a, b]$.

Definition 2.5. [8] Let $f : [a, b] \to \mathbf{E}^n$. We call f is measurable, if for each $\alpha \in [0, 1]$, the set-valued mapping $[f]^\alpha : [a, b] \to P_{kc}(\mathbf{R}^n)$ is measurable, where $[f]^\alpha$ is the α-level cut of f, $P_{kc}(\mathbf{R}^n)$ be the set of nonempty compact and convex subset of $\mathbf{R}^n$.

Definition 2.6. [8] Let $f : [a, b] \to \mathbf{E}^n$. We call f is integrably bounded, if there exists a Lebesgue integrable function $h(t)$, such that for each $x \in [f]^0$, $||x||_2 \leq h(t)$.

Definition 2.7. [8] Let $f : [a, b] \to \mathbf{E}^n$ is integrably bounded, for all measurable subset $A \in [a, b]$, define

$$[\int_A f(t)dt]^\alpha = \int_A [f]^\alpha dt$$
$$= \{\int_A g(t)dt \mid g(t) \in [f]^\alpha \text{ be an integrable selector of } [f]^\alpha\}$$

if there exists $u \in \mathbf{E}^n$, such that $[u]^\alpha = [\int_A f(t)dt]^\alpha$, $\alpha \in [0, 1]$, then we call f is integrably on $[a, b]$, and $\int_A f(t)dt = u$.

Proposition 2.1. *[8] Let $f : [a, b] \to \mathbf{E}^n$ be continuous, then f is integrable on $[a, b]$.*

Proposition 2.2. *[8] Let $f : [a, b] \to \mathbf{E}^n$ be integrable on $[a, b]$, $a \le c \le b$, then f is integrable on $[a, c]$, $[c, b]$ and*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Theorem 2.3. *[8] Let $f : [a, b] \to \mathbf{E}^n$ be continuous on $[a, b]$, then*

$$\left(\int_a^t f(\tau)d\tau\right)' = f(t),\ t \in [a, b].$$

3. The existence of periodic solutions

In this section, the following periodic problem for first-order uncertain dynamical system will be studied:

$$\begin{cases} \xi'(t) \in f(t, \xi(t)), \\ \xi(0) = \xi(T) \end{cases} \tag{3.1}$$

where $t \in I \triangleq [0, T] \subset \mathbf{R}$, $f : I \times \mathbf{R}^n \to \mathbf{E}^n$. By taking the α-cut of (3.1), the following class of differential inclusions are taken into consideration.

$$\xi'(t) \in F(t, \xi(t), \alpha),\ \ \xi(0) = \xi(T)\ (\alpha \in [0, 1]), \tag{3.2}$$

denote $F(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$.

Like [9], the definitions of solutions to (3.1) and (3.2) are given as follows.

Definition 3.1. Define $\xi(t)$ be a solution of (3.2) for any fixed $\alpha \in [0, 1]$, if $\xi(t)$ satisfies the following conditions:

(1) $\xi(t)$ is absolutely continuous on I;
(2) $\xi'(t) \in F(t, \xi(t), \alpha)$ a.e. on I;
(3) $\xi(0) = \xi(T)$.

Definition 3.2. For $\alpha \in [0, 1]$, define $\Sigma_\alpha(I; t) = \{\xi(t) | \xi(t) \text{ is a solution of } (3.2)\}$ be the set of solutions of (3.2).

Definition 3.3. Define $v : I \to \mathbf{D}^n$ be the solution of (3.1), if $v(t)$ satisfies: $[v(t)]^\alpha = \Sigma_\alpha(I; t)$ $(t \in I,\ 0 \le \alpha \le 1)$.

Before solving the problem (3.1), some definitions and lemmas are needed.

Let $W^{1,1}([a, b], \mathbf{R}^n)$ be the Sobolev space with the norm $||x||_{W^{1,1}} = \int_a^b ||x(t)||_2 dt + \int_a^b ||x'(t)||_2 dt$ for $x \in W^{1,1}([a, b], \mathbf{R}^n)$, then $W^{1,1}([a, b], \mathbf{R}^n)$ is a Banach space and $W^{1,1}([a, b], \mathbf{R}^n)$ can be compactly embedded into $L^1([a, b], \mathbf{R}^n)$ (see [26]).

Lemma 3.1. *[26] Let $AC([a,b],\mathbf{R}^n)$ be the set of absolutely continuous functions on $[a,b]$, then $x \in AC([a,b],\mathbf{R}^n)$ if and only if $x \in W^{1,1}([a,b],\mathbf{R}^n)$.*

Lemma 3.2. *[24] Let $W_p^{1,1}(I,\mathbf{R}^n) = \{\xi \in W^{1,1}(I,\mathbf{R}^n)|\ \xi(0) = \xi(T)\}$. If $L : W_p^{1,1}(I,\mathbf{R}^n) \to L^1(I,\mathbf{R}^n)$ is defined as $L(\xi) = \xi' - \xi$, then L is invertible, and $L^{-1} : L^1(I,\mathbf{R}^n) \to L^1(I,\mathbf{R}^n)$ is a compact operator.*

Definition 3.4. [4] Let Y and Z be Hausdorff topological spaces. We say that the set-valued mapping $G : Y \to 2^Z \setminus \{\phi\}$ is upper semicontinuous, if for any nonempty closed subset C of Z, the set $G^-(C) = \{y \in Y|\ G(y) \cap C \neq \phi\}$ is a closed subset of Y.

For a Banach space Y, let $P_{kc}(Y)$ and $P_{wk}(Y)$ be the set of nonempty compact and convex subsets of Y and the set of nonempty weakly compact subsets of Y, respectively. Let (Ω, Σ, μ) be a measure space and $L^p(\Omega, Y)$ be the space of $p-$Bochner integrable functions ($p \geq 1$).

Lemma 3.3. *[28] Let $\{f_n\}_{n=1}^{\infty} \subset L^p(\Omega, Y),\ f \in L^p(\Omega, Y),\ f_n \xrightarrow{w} f$ and $f_n(x) \in G(x)$ $\mu-$ a.e. on Ω, where $G(x) \in P_{wk}(Y)$ $\mu-$ a.e. on Ω. Then $f(x) \in \overline{conv}(w - \overline{\lim}\{f_n(x)\}_{n\geq 1})$ $\mu-$ a.e. on Ω.*

Lemma 3.4. *[18] Let Y be a Banach space, C be a nonempty closed and convex subset of Y and the null element $\theta \in C$. If $G : C \to P_{kc}(C)$ is upper semicontinuous and maps any bounded set into sequentially compact set, then either the set $J = \{x \in C|\ x \in \lambda G(x),\ \lambda \in (0,1)\}$ is unbounded or the set-valued mapping G admits a fixed point, i.e. there exists $x \in C$ such that $x \in G(x)$.*

Theorem 3.1. *Suppose that $f : I \times \mathbf{R}^n \to \mathbf{E}^n$ satisfies:*

(i) *$f \in C(I \times \mathbf{R}^n, \mathbf{E}^n)$, i.e. f is continuous on $I \times \mathbf{R}^n$.*
(ii) *There exist $\beta > 0,\ \alpha(t) \in L^1(I, {\mathbf{R}^n}_+)$ such that*

$$D(f(t,\xi),\hat{0}) \leq \alpha(t) + \beta||\xi||_2 \ \text{ a.e. on } I.$$

(iii) *There exists $G > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) > 0,\ m(\xi_0) > 0$ such that*

$$\inf\{\langle \xi, \zeta\rangle | \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \geq m(\xi_0),$$

for a.e. $t \in I$.

Then the set of solutions $\sum_\alpha(I;t)$ to (3.2) is nonempty and $\sum_\alpha(I;t)$ is uniformly bounded $(0 \leq \alpha \leq 1)$.

Proof. By Lemma 3.1-3.4, and similar to the proof of [24] and Theorem 3.1-3.2 of [11], the above conclusion can be concluded. □

Lemma 3.5. *[29] Let $\mathbf{B}$ be a separable normed linear space. Then any bounded set in $\mathbf{B}^*$ is weakly$*$ sequentially compact.*

Lemma 3.6. *[15] Let Ω be a open set of $\mathbf{R} \times \mathbf{R}^n$, $f : \Omega \to \mathbf{E}^n$ is upper semicontinuous, and let $F(t,\xi(t),\alpha) = [f(t,\xi)]^\alpha :\ \Omega \times [0,1] \to P_{kc}(\mathbf{R}^n)$. Then $F(t,\xi(t),\alpha)$ is upper semicontinuous in $\Omega \times [0,1]$.*

Lemma 3.7 *(Convergence Theorem). [4] Let F be a proper semicontinuous map from a Hausdorff locally convex space X to the closed convex subsets of a Banach space Y. Let I be an interval of $\mathbf{R}$ and x_k and y_k be measurable functions from I to X and Y respectively satisfying: for almost all $t \in I$, for every neighborhood N of θ in $X \times Y$, there exists $k_0 = k_0(t,N)$ such that $\forall k \geq k_0$, $(x_k(t), y_k(t)) \in graph(F) + N$. If*

i) *$x_k(\cdot)$ converges almost everywhere to a function $x(\cdot)$ from I to X,*

ii) *$y_k(\cdot)$ belongs to $L^1(I,Y)$ and converges weakly to $y(\cdot)$ in $L^1(I,Y)$,*

then for almost all $t \in I$, $(x(t), y(t)) \in graph(F)$, i.e., $y(t) \in F(x(t))$.

Theorem 3.2. *Suppose that $f : I \times \mathbf{R}^n \to \mathbf{E}^n$ satisfies:*

(i) $f \in C(I \times \mathbf{R}^n, \mathbf{E}^n)$.
(ii) *There exist $\beta > 0$, $\alpha(t) \in L^1(I, \mathbf{R}^n{}_+)$ such that*

$$D(f(t,\xi), \hat{0}) \leq \alpha(t) + \beta ||\xi||_2 \text{ a.e. on } I.$$

(iii) *There exists $G > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) > 0$, $m(\xi_0) > 0$ such that*

$$\inf\{\langle \xi, \zeta\rangle \mid \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \geq m(\xi_0),$$

for a.e. $t \in I$.
Then there exists a solution $v : I \to \mathbf{D}^n$ of (3.1) such that $[v(t)]^\alpha = \sum_\alpha(I;t)$ $(t \in I)$, $\alpha \in [0,1]$, and $v(0) = v(T)$.

Proof. From the definition of the solution to (3.1) (Definition 3.3), the existence of $\Sigma_\alpha(I;t)$ to (3.2) should be proved first. By Theorem 3.1, $\Sigma_\alpha(I;t) \neq \phi$ $(0 \leq \alpha \leq 1)$. Denote $\Sigma_\alpha = \Sigma_\alpha(I;t)$ for simplicity.

Then, the existence of solution $v : I \to \mathbf{D}^n$ such that $[v(t)]^\alpha = \Sigma_\alpha(I;t)$ $(t \in I,\ 0 \leq \alpha \leq 1)$ will be proved. The Theorem 2.1 (Stacking Theorem) is used in this case. $\Sigma_\alpha \neq \phi$ $(0 \leq \alpha \leq 1)$ has been gotten already. Next, Σ_α $(0 \leq \alpha \leq 1)$ are compact sets should be proved.

For $\forall \xi \in \Sigma_\alpha$ be a solution to (3.2), then

$$\xi'(t) \in F(t, \xi(t), \alpha) \subset F(t, \xi(t), 0).$$

By Theorem 3.1, $\{\xi(t) | \xi \in \Sigma_\alpha\}$ is uniformly bounded on I. Then there exists $M > 0$, such that $\forall \xi \in \sum_\alpha(I;\cdot)$, $||\xi(t)||_2 \leq M (t \in I)$, $\alpha \in [0,1]$. As $\xi'(t) \in F(t,\xi(t),\alpha)$ a.e. on I and f is continuous on $[0,T] \times [-M, M]$, there exists $M' > 0$ such that $||\xi'(t)||_2 \leq M'$ a.e. on I $(0 \leq \alpha \leq 1)$.

By the definition of solution to (3.2), $\{\xi(t) | \xi \in \Sigma_\alpha\}$ is absolutely continuous on I, then $\{\xi(t) | \xi \in \Sigma_\alpha\}$ is equicontinuous on I.

It can be concluded that $\{\xi(t) | \xi \in \Sigma_\alpha\}$ is uniformly bounded and equicontinuous on I and $\{\xi'(t) | \xi \in \Sigma_\alpha\}$ is bounded in $L^\infty(I, \mathbf{R}^n)$.

As $L^1(I, \mathbf{R}^n)$ is a separable Banach space and $(L^1(I, \mathbf{R}^n))^* = L^\infty(I, \mathbf{R}^n)$, by Lemma 3.5 the set $\{\xi'(\cdot) | \xi \in \Sigma_\alpha\} \subset L^\infty(I, \mathbf{R}^n)$ is weakly$*$ sequentially compact. Then, arbitrarily choose $\{\xi_n\} \subset \Sigma_\alpha$, there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ such that $\xi'_{n_k} \overset{weakly^*}{\longrightarrow} \zeta \in L^\infty(I, \mathbf{R}^n)$, i.e., $\forall h \in L^1(I, \mathbf{R}^n)$, have

$$\int_0^T \langle h(t), \xi'_{n_k}(t)\rangle dt \to \int_0^T \langle h(t), \zeta(t)\rangle dt.$$

Suitably choosing $h \in L^1(I, \mathbf{R}^n)$, have

$$\int_0^t \xi'_{n_k}(s) ds \to \int_0^t \zeta(s) ds.$$

And by Ascoli-Arzelá Theorem, without loss of generality, assume that there exists $\xi \in C(I, \mathbf{R}^n)$ such that

$$\max_{t \in I} ||\xi_{n_k}(t) - \xi(t)||_2 \to 0.$$

As $\xi_{n_k}(t)$ is absolutely continuous $(k = 1, 2, \cdots)$, then $\xi_{n_k}(t) - \xi_{n_k}(0) \to \int_0^t \zeta(s)ds$ and further

$$\xi_{n_k}(t) \to \xi(0) + \int_0^t \zeta(s)ds.$$

So $\xi(t) = \xi(0) + \int_0^t \zeta(s)ds$, i.e., $\xi(t)$ is absolutely continuous and $\zeta = \xi' \in L^\infty(I, \mathbf{R}^n)$. Then $\forall h \in L^1(I, \mathbf{R}^n)$, have

$$\int_0^T \langle h(t), \xi'_{n_k}(t)\rangle dt \to \int_0^T \langle h(t), \xi'(t)\rangle dt.$$

As $\xi'_{n_k}(t) \in F(t, \xi_{n_k}(t), \alpha)$ a.e. on I, by the continuity of f and Proposition 2.1-2.2, $\forall [a, b] \subset I$, then

$$\int_a^b \xi'_{n_k}(t)dt \in \int_a^b F(t, \xi_{n_k}(t), \alpha)dt \quad (k = 1, 2, \cdots).$$

After suitably choosing $h \in L^1(I, \mathbf{R}^n)$, it can be gotten that $\int_a^b \xi'_{n_k}(t)dt \to \int_a^b \xi'(t)dt$. Therefore, by the continuity of f, have

$$\int_a^b \xi'(t)dt \in \int_a^b F(t, \xi(t), \alpha)dt.$$

Then for $t \in I,\ t + \Delta t \in I\ (\Delta t > 0)$, have

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} \xi'(s)ds \in \frac{1}{\Delta t}\int_t^{t+\Delta t} F(s, \xi(s), \alpha)ds.$$

By the absolute continuity of $\xi(t)$, the continuity of f and Theorem 2.3, letting $\Delta t \to 0$, then

$$\xi'(t) \in F(t, \xi(t), \alpha) \quad \text{a.e. on } \ I.$$

From $\xi_{n_k}(0) = \xi_{n_k}(T)$, it is immediate that $\xi(0) = \xi(T)$. Therefore, $\xi \in \Sigma_\alpha$.

Then from the argument above there exists a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\} \subset \Sigma_\alpha$ such that $\exists \xi \in \Sigma_\alpha$ satisfying

$$\max_{t\in I} ||\xi_{n_k}(t) - \xi(t)||_2 \to 0, \quad \text{and} \quad \xi'_{n_k} \overset{weakly^*}{\longrightarrow} \xi'.$$

As $L^1(I, \mathbf{R}^n)$ is separable, without loss of generality, assume that there exists $\{h_k\} \subset L^1(I, \mathbf{R}^n)$ such that $\overline{\{h_k\}} = L^1(I, \mathbf{R}^n)$ and $h_k \neq 0$ $(k = 1, 2, \cdots)$. Denote $\mathcal{A} \overset{\triangle}{=} \{\xi(\cdot) \in C(I, \mathbf{R}^n) | \xi'(\cdot) \in L^\infty(I, \mathbf{R}^n)\}$. For $\mathcal{A}$, introduce the following norm:

$$||\xi||^* = \max_{t\in I} ||\xi(t)||_2 + \sum_{k=1}^{\infty} \frac{1}{2^k ||h_k||_{L^1}} \left| \int_0^T \langle h_k(t), \xi'(t)\rangle dt \right|, \quad \forall \xi \in \mathcal{A},$$

then it is easy to verify that $(\mathcal{A},\ ||\cdot||^*)$ is a normed linear space and Σ_α is a bounded subset of $\mathcal{A}$. Moreover it is obvious that for $\{\xi_n\} \subset \mathcal{A},\ \xi \in \mathcal{A}$ and $\{\xi'_n\}$ bounded in $L^\infty(I, \mathbf{R}^n)$, then $||\xi_n - \xi||^* \to 0$ if and only if $\max_{t\in I} ||\xi_n(t) - \xi(t)||_2 \to 0$ and $\xi'_n \overset{weakly^*}{\longrightarrow} \xi'$.

So it can be concluded that Σ_α is a compact subset of $\mathcal{A}$.

Finally, to proof that Σ_α satisfy the condition (i) and (ii) of the Theorem 2.1 (Stacking Theorem).

As $f : I \times \mathbf{R}^n \to \mathbf{E}^n$, it is obvious that $\Sigma_\beta \subset \Sigma_\alpha$ $(0 \le \alpha \le \beta \le 1)$.

Let $\{\alpha_n\} \subset [0,1]$ be monotone increasing and $\alpha_n \to \alpha \in [0,1]$, then it is easy to get that $\Sigma_\alpha \subset \bigcap_{n=1}^{\infty} \Sigma_{\alpha_n}$. For $\forall \xi^*(t) \in \bigcap_{n=1}^{\infty} \Sigma_{\alpha_n}$, we'll prove that $\xi^*(t) \in \Sigma_\alpha$. As the graph of F: $Gr(F)$ is compact, then $\inf\{||a-b||_2 : a \in Gr(F),\ b \in (Gr(F)+N)^c\} > \varepsilon > 0$, where $\varepsilon > 0$ is sufficiently small, N is the neighborhood of null element θ in $\Omega \times \mathbf{R}^n \times [0,\ 1]$ and Ω is the open set of $\mathbf{R} \times \mathbf{R}^n$. As $F(t,\xi,\alpha) = [f(t,\xi)]^\alpha$ is upper semicontinuous, then there exists U is the neighborhood of $(t,\xi^*(t),\alpha)$, have $F(s,\xi(s),\beta) \subset F(t,\xi^*(t),\alpha) + \varepsilon B$ for $\forall (s,\xi(s),\beta) \in U$, where B is the unit ball in $\mathbf{R}^n$. Let n sufficiently large, then $(t,\xi^*(t),\alpha_n) \in U$, we have $F(t,\xi^*(t),\alpha_n) \subset F(t,\xi^*(t),\alpha) + \varepsilon B$. That is to say $(t,\xi^*(t),\alpha_n,\xi^{*\prime}(t)) \in Gr(F) + N$. By Lemma 3.7 (Convergence Theorem), we have $\xi^{*\prime}(t) \in F(t,\xi^*(t),\alpha)$. And as $\xi^*(t) \in \bigcap_{n=1}^{\infty} \Sigma_{\alpha_n}$, then $\xi^*(0) = \xi^*(T)$. So $\xi^*(t) \in \Sigma_\alpha$. Then we have $\Sigma_\alpha = \bigcap_{n=1}^{\infty} \Sigma_{\alpha_n}$.

Therefore by the Theorem 2.1 (Stacking Theorem), there exists unique $v : I \to \mathbf{D}^n$ such that $[v(t)]^\alpha = \Sigma_\alpha$ $(0 \le \alpha \le 1), t \in I$. Then there exists a unique solution $v(t)$ $(t \in I)$ of (3.1). □

Example 3.1. For (3.1), take $I = [0,1]$, $f(t,\xi) = \mathrm{e}^t \otimes u_0 + \xi$, $u_0 \in \mathbf{E}^1$ and $[u_0]^\alpha = [1+\alpha,\ 3-\alpha]$, $\alpha \in [0,1]$. Then f satisfies the conditions (i), (ii) in Theorem 3.2. And for condition (iii), there exists $G = 1.5 > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) = 0.4 > 0$, $m(\xi_0) = 0.11 > 0$ such that

$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$

for a.e. $t \in I$. So there exists the unique periodic solution $v : [0,1] \to \mathbf{D}^1$ to (3.1) with $v(0) = v(1)$, $v(t) = \mathrm{e}^t(t + \frac{\mathrm{e}}{1-\mathrm{e}}) \otimes u_0$, with $v(0) = v(1) = \frac{\mathrm{e}}{1-\mathrm{e}} \otimes u_0$, where "$\otimes$" is the operation of product based on Zadeh's Extension Principle.

Example 3.2. For (3.1), take $I = [0,1]$, $f(t,\xi) = \mathrm{e}^{-t} \otimes u + \xi$, $u \in \mathbf{E}^n$. Then f satisfies the conditions (i), (ii) in Theorem 3.2. And for condition (iii), let $l = \inf\{||\mu||_2 \mid \mu \in [u]^0\}$, there exists $G = \frac{l+2}{\mathrm{e}} > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) = \frac{1}{\mathrm{e}} > 0$, $m(\xi_0) = \frac{l+1}{\mathrm{e}^2} > 0$ such that

$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$

for a.e. $t \in I$. So there exists the unique periodic solution $v : [0,1] \to \mathbf{D}^n$ to (3.1) with $v(0) = v(1)$, where $v(t) = -\frac{1}{2}\mathrm{e}^t(\mathrm{e}^{-1} + \mathrm{e}^{-2t}) \otimes u$, with $v(0) = v(1) = -\frac{1}{2}(1 + \mathrm{e}^{-1}) \otimes u$.

4. The structural stability of periodic solutions

In section 3, the existence of periodic solutions for first-order uncertain dynamical system has been proved on $\mathbf{E}^n$. The structural stability of periodic solutions is also an interesting property that we are concerned about. Next we will discuss structure to the first-order uncertain dynamical system if given a specific perturbation to the forcing function as follows.

$$\begin{cases} \xi'(t) \in f_k(t,\xi(t)), \\ \xi(0) = \xi(T) \end{cases} \tag{4.1}$$

where $t \in I \triangleq [0,T] \subset \mathbf{R}$, and $f_k : I \times \mathbf{R}^n \to \mathbf{E}^n$. By taking the α-cut of (4.1), it can be considered as a class of differential inclusions:

$$\xi'(t) \in [f_k(t,\xi(t))]^\alpha,\ \xi(0) = \xi(T)\ (\alpha \in [0,1]). \tag{4.2}$$

Theorem 4.1. *Suppose that $f,\ f_k : I \times \mathbf{R}^n \to \mathbf{E}^n$ satisfies (i), (ii) and (iii) in Theorem 3.2, and (iv) $\lim_{k\to\infty} D(f_k, f) = 0$.*

Then the solution $v_k : I \to \mathbf{D}^n$ to (4.1) and the solution $v : I \to \mathbf{D}^n$ to (3.1) satisfy $\lim_{k\to\infty} D(v_k, v) = 0$ for $t \in I$.

Proof. By the Theorem 3.2, the existences of solutions $v_k, v : I \to \mathbf{D}^n$ such that $[v_k(t)]^\alpha = \Sigma_\alpha^k(I;t)$, $[v(t)]^\alpha = \Sigma_\alpha(I;t)$ $(t \in I,\ 0 \le \alpha \le 1)$ are obvious. Denote $\Sigma_\alpha^k = \Sigma_\alpha^k(I;t)$, $\Sigma_\alpha = \Sigma_\alpha(I;t)$ for simplicity.

From (iv), $\forall\varepsilon > 0$, there exists K, when $k > K$, we have $D(f_k, f) < \frac{1}{2}\varepsilon$, i.e., $\sup\limits_{0\le\alpha\le 1} H(F_k, F) < \frac{1}{2}\varepsilon$. By the definition of Hausdorff metric H, $H(F_k, F) = \max\{\rho(F_k, F), \rho(F, F_k)\}$, where $\rho(F_k, F) = \inf\{\varepsilon > 0 \mid F_k \subset N(F, \varepsilon)\} = \inf\{\varepsilon > 0 \mid F_k \subset F + \varepsilon B\}$, $\rho(F, F_k) = \inf\{\varepsilon > 0 \mid F \subset N(F_k, \varepsilon)\} = \inf\{\varepsilon > 0 \mid F \subset F_k + \varepsilon B\}$. Therefore $\forall\varepsilon > 0$, there exists K, when $k > K$, we have

$$F_k(t, \xi, \alpha) \subset F(t, \xi, \alpha) + \frac{1}{2}\varepsilon B$$
$$F(t, \xi, \alpha) \subset F_k(t, \xi, \alpha) + \frac{1}{2}\varepsilon B$$

$\forall\alpha \in [0, 1],\ t \in I$, where $F(t, \xi, \alpha),\ F_k(t, \xi, \alpha) \in P_k(\mathbf{R}^n)$.

For $\forall\xi^*(t) \in \lim\limits_{k\to\infty} \Sigma_\alpha^k$, there exists $\xi_k(t) \in \Sigma_\alpha^k$ with $\lim\limits_{k\to\infty} \xi_k(t) = \xi^*(t)$ for $\forall t \in I$. And we have

$$\begin{cases} \xi_k{}'(t) \in F_k(t, \xi_k(t), \alpha) \in F(t, \xi_k(t), \alpha) + \frac{1}{2}\varepsilon B, \\ \xi_k(0) = \xi_k(T) \end{cases}$$

Next we'll prove that $\xi^*(t) \in \Sigma_\alpha$. As $F \in P_k(\mathbf{R}^n)$, then the graph of F: $Gr(F)$ is compact and $\inf\{||a - b||_2 : a \in Gr(F),\ b \in (Gr(F) + N)^c\} > \varepsilon > 0$, where $\varepsilon > 0$ is sufficiently small, N is the neighborhood of null element θ in $\Omega \times \mathbf{R}^n \times [0,\ 1]$ and Ω is the open set of $\mathbf{R} \times \mathbf{R}^n$. And as $F(t, \xi, \alpha) = [f(t, \xi)]^\alpha$ is upper semicontinuous, then there exists U is the neighborhood of $(t, \xi^*(t), \alpha)$, have $F(s, \xi(s), \beta) \subset F(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon B$ for $\forall(s, \xi(s), \beta) \in U$, where B is the unit ball in $\mathbf{R}^n$. Let k sufficiently large, then $(t, \xi_k(t), \alpha) \in U$ and $F(t, \xi_k(t), \alpha) \subset F(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon B$. And then $F_k(t, \xi_k(t), \alpha) \subset F(t, \xi^*(t), \alpha) + \varepsilon B$. That is to say $(t, \xi_k(t), \alpha, \xi_k{}'(t)) \in Gr(F) + N$. By Lemma 3.7 (Convergence Theorem), we have $\xi^{*\prime}(t) \in F(t, \xi^*(t), \alpha)$. And as $\xi^*(t) \in \lim\limits_{k\to\infty} \Sigma_\alpha^k$, then $\xi^*(0) = \xi^*(T)$. So $\xi^*(t) \in \Sigma_\alpha$. Then we have $\lim\limits_{k\to\infty} \Sigma_\alpha^k \subset \Sigma_\alpha$.

For $\forall\xi^*(t) \in \Sigma_\alpha$, we have

$$\begin{cases} \xi^{*\prime}(t) \in F(t, \xi^*(t), \alpha) \subset F_k(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon B, \\ \xi^*(0) = \xi^*(T) \end{cases}$$

For $\xi^*(t)$, there exists $(t, \xi_{\delta_k}(t), \alpha)$ in the neighborhood U of $(t, \xi^*(t), \alpha)$, have $|\xi_{\delta_k}(t) - \xi^*(t)| < \delta_k$ for $t \in I$ and $\xi_{\delta_k}(0) = \xi_{\delta_k}(T)$. And then $(t, \xi^*(t), \alpha)$ is also in the neighborhood U_{δ_k} of $(t, \xi_{\delta_k}(t), \alpha)$. As $F_k(t, \xi, \alpha) = [f_k(t, \xi)]^\alpha$ is upper semicontinuous, then $F_k(t, \xi^*(t), \alpha) \subset F_k(t, \xi_{\delta_k}(t), \alpha) + \frac{1}{2}\varepsilon B$. That is to say $(t, \xi^*(t), \alpha, \xi^{*\prime}(t)) \in Gr(F_k(t, \xi_{\delta_k}(t), \alpha)) + N$. By Lemma 3.7 (Convergence Theorem), we have $\xi_{\delta_k}{}'(t) \in F_k(t, \xi_{\delta_k}(t), \alpha)$. And it's obvious that $\xi_{\delta_k}(t)$ is absolutely continuous. So $\xi_{\delta_k}(t) \in \Sigma_\alpha^k$. Then we have $\Sigma_\alpha \subset \lim\limits_{k\to\infty} \Sigma_\alpha^k$. Therefore, $\Sigma_\alpha = \lim\limits_{k\to\infty} \Sigma_\alpha^k$.

Then $\lim\limits_{k\to\infty} D(v_k, v) = \lim\limits_{k\to\infty} \sup\limits_{0\le\alpha\le 1} H(\Sigma_\alpha^k, \Sigma_\alpha) = 0.$ □

Theorem 4.1 has proved that the structure stability of first-order uncertain dynamical systems if the perturbation forcing functions converges. And if the perturbation forcing functions are also monotone, then a better result could be obtained. Before that, the definition of monotone fuzzy numbers is given as follows.

For $u, v \in \mathbf{D}^n$, $u \subset v$ if and only if $[u]^\alpha \subset [v]^\alpha$ $(\forall\alpha \in [0, 1])$. For $\{u_n\} \subset \mathbf{D}^n$ is monotone, if $u_{n+1} \subset u_n$ or $u_n \subset u_{n+1}$, $(n = 1, 2, ...)$ (see [32]).

Theorem 4.2. *Suppose that $f,\ f_k : I \times \mathbf{R}^n \to \mathbf{E}^n$ satisfies (i), (ii) and (iii) in Theorem 3.2, and (iv) f_k is monotone, and $\lim\limits_{k\to\infty} D(f_k, f) = 0$.*

Then the solution $v_k : I \to \mathbf{D}^n$ to (4.1) and the solution $v : I \to \mathbf{D}^n$ to (3.1) satisfy $\lim\limits_{k\to\infty} D(v_k, v) = 0$ uniformly for $t \in I$.

Proof. By the Theorem 3.2, the existences of solutions $v_k, v : I \to \mathbf{D}^n$ such that $[v_k(t)]^\alpha = \Sigma_\alpha^k(I; t),\ [v(t)]^\alpha = \Sigma_\alpha(I; t)$ $(t \in I,\ 0 \le \alpha \le 1)$ are obvious. Denote $\Sigma_\alpha^k = \Sigma_\alpha^k(I; t)$, $\Sigma_\alpha = \Sigma_\alpha(I; t)$ for simplicity.

As f_k is monotone, two situations will be discussed to prove that $\lim\limits_{k\to\infty} D(v_k, v) = 0$ uniformly for $t \in I$.

(1) If f_k is monotone decreasing, i.e., $f_{k+1} \subset f_k$, $(k = 1, 2, ...)$.
From (iv), we have

$$F(t, \xi, \alpha) \subset \cdots \subset F_{k+1}(t, \xi, \alpha) \subset F_k(t, \xi, \alpha) \subset \cdots \subset F_1(t, \xi, \alpha),$$

$\forall \alpha \in [0, 1]$, where $F(t, \xi, \alpha),\ F_k(t, \xi, \alpha) \in P_k(\mathbf{R}^n)$.
For $\forall \xi^*(t) \in \Sigma_\alpha$, we have

$$\begin{cases} \xi^{*\prime}(t) \in F(t, \xi^*(t), \alpha) \subset \cdots \subset F_k(t, \xi^*(t), \alpha) \subset \cdots \subset F_1(t, \xi^*(t), \alpha), \\ \xi^*(0) = \xi^*(T) \end{cases}$$

then $\xi^*(t) \in \Sigma_\alpha^k$ and $\Sigma_\alpha \subset \cdots \subset \Sigma_\alpha^k \subset \cdots \subset \Sigma_\alpha^1$. So it is obvious that $\Sigma_\alpha \subset \bigcap\limits_{k=1}^{\infty} \Sigma_\alpha^k$.

For $\forall \xi^*(t) \in \bigcap\limits_{k=1}^{\infty} \Sigma_\alpha^k$, there exists $\xi_k(t) \in \Sigma_\alpha^k$ with $\lim\limits_{k\to\infty} \xi_k(t) = \xi^*(t)$ for $\forall t \in I$. As $\lim\limits_{k\to\infty} D(f_k, f) = 0$ uniformly for $t \in I$, then

$$H(F_k(t, \xi, \alpha), F(t, \xi, \alpha)) \to 0,$$

for $\forall t \in I$, $\forall \alpha \in [0, 1]$.

As the graph of F: $Gr(F)$ is compact, then $\inf\{||a - b||_2 : a \in Gr(F),\ b \in (Gr(F) + N)^c\} > \varepsilon > 0$, where $\varepsilon > 0$ is sufficiently small, N is the neighborhood of null element θ in $\Omega \times \mathbf{R}^n \times [0,\ 1]$ and Ω is the open set of $\mathbf{R} \times \mathbf{R}^n$. As $F(t, \xi, \alpha) = [f(t, \xi)]^\alpha$ is upper semicontinuous, then there exists U is the neighborhood of $(t, \xi^*(t), \alpha)$, have $F(s, \xi(s), \beta) \subset F(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon B$ for $\forall (s, \xi(s), \beta) \in U$, where B is the unit ball in $\mathbf{R}^n$. Let k sufficiently large, then $(t, \xi_k(t), \alpha) \in U$ and $F_k(t, \xi_k(t), \alpha) \subset F(t, \xi_k(t), \alpha) + \frac{1}{2}\varepsilon B$. And then $F_k(t, \xi_k(t), \alpha) \subset F(t, \xi^*(t), \alpha) + \varepsilon B$. That is to say $(t, \xi_k(t), \alpha, \xi_k{}'(t)) \in Gr(F) + N$. By Lemma 3.7 (Convergence Theorem), we have $\xi^{*\prime}(t) \in F(t, \xi^*(t), \alpha)$. And as $\xi^*(t) \in \bigcap\limits_{k=1}^{\infty} \Sigma_\alpha^k$, then $\xi^*(0) = \xi^*(T)$. So $\xi^*(t) \in \Sigma_\alpha$. Then we have $\Sigma_\alpha = \bigcap\limits_{k=1}^{\infty} \Sigma_\alpha^k$.

Let $\varphi_k(x) = \sigma_{\Sigma_\alpha^k}(x)$, $\varphi(x) = \sigma_{\Sigma_\alpha}(x)$, $x \in S^{n-1}$, then $\varphi_k(x)$, $\varphi(x)$ are continuous with respect to x, and $\lim\limits_{k\to\infty} \varphi_k(x) = \varphi(x)$. For each fixed x, $\varphi_k(x)$ is monotonously decreasing, then by Dini Theorem, $\lim\limits_{k\to\infty} \varphi_k(x) = \varphi(x)$ uniformly with respect to $x \in S^{n-1}$.

Therefore,

$$H(\Sigma_\alpha^k, \Sigma_\alpha) = \sup_{x \in S^{n-1}} \{|\sigma_{\Sigma_\alpha^k}(x) - \sigma_{\Sigma_\alpha}(x)| :\ x \in S^{n-1}\} \to 0,\ k \to \infty$$

uniformly for $\alpha \in [0, 1]$.

Then $\lim\limits_{k\to\infty} D(v_k, v) = \lim\limits_{k\to\infty} \sup\limits_{0\le\alpha\le1} H(\Sigma_\alpha^k, \Sigma_\alpha) = 0$, uniformly for $t \in I$.

(2) If f_k is monotone increasing, i.e., $f_k \subset f_{k+1}$, $(k = 1, 2, ...)$.
From (iv), we have

$$F_1(t, \xi, \alpha) \subset \cdots \subset F_k(t, \xi, \alpha) \subset F_{k+1}(t, \xi, \alpha) \subset \cdots \subset F(t, \xi, \alpha),$$

$\forall \alpha \in [0, 1]$, where $F(t, \xi, \alpha),\ F_k(t, \xi, \alpha) \in P_k(\mathbf{R}^n)$.
As

$$\begin{cases} \xi'(t) \in F_1(t, \xi(t), \alpha) \subset \cdots \subset F_k(t, \xi(t), \alpha) \subset \cdots \subset F(t, \xi(t), \alpha), \\ \xi(0) = \xi(T) \end{cases}$$

then $\Sigma_\alpha^1 \subset \cdots \subset \Sigma_\alpha^k \subset \cdots \subset \Sigma_\alpha$.

For $\forall \xi^*(t) \in \bigcup_{k=1}^{\infty} \Sigma_\alpha^k$, there exists $\xi_k(t) \in \Sigma_\alpha^k$ with $\lim_{k\to\infty} \xi_k(t) = \xi^*(t)$ for $\forall t \in I$. As $\lim_{k\to\infty} D(f_k, f) = 0$ uniformly for $t \in I$, then

$$H(F_k(t,\xi,\alpha), F(t,\xi,\alpha)) \to 0,$$

for $\forall t \in I$, $\forall \alpha \in [0, 1]$.

As the graph of F: $Gr(F)$ is compact, then $\inf\{||a-b||_2 : a \in Gr(F),\ b \in (Gr(F)+N)^c\} > \varepsilon > 0$, where $\varepsilon > 0$ is sufficiently small, N is the neighborhood of θ in $\Omega \times \mathbf{R}^n \times [0,\ 1]$ and Ω is the open set of $\mathbf{R} \times \mathbf{R}^n$. As $F(t,\xi,\alpha) = [f(t,\xi)]^\alpha$ is upper semicontinuous, then there exists U is the neighborhood of $(t, \xi^*(t), \alpha)$, have $F(s, \xi(s), \beta) \subset F(t, \xi^*(t), \alpha) + \varepsilon B$ for $\forall (s, \xi(s), \beta) \in U$, where B is the unit ball in $\mathbf{R}^n$. Let k sufficiently large, then $(t, \xi_k(t), \alpha) \in U$ and $F_k(t, \xi_k(t), \alpha) \subset F(t, \xi_k(t), \alpha)$. And then $F_k(t, \xi_k(t), \alpha) \subset F(t, \xi^*(t), \alpha) + \varepsilon B$. That is to say $(t, \xi_k(t), \alpha, \xi_k{}'(t)) \in Gr(F) + N$. By Lemma 3.7 (Convergence Theorem), we have $\xi^{*\prime}(t) \in F(t, \xi^*(t), \alpha)$. And as $\xi^*(t) \in \bigcup_{k=1}^{\infty} \Sigma_\alpha^k$, then $\xi^*(0) = \xi^*(T)$. So $\xi^*(t) \in \Sigma_\alpha$. So $\bigcup_{k=1}^{\infty} \Sigma_\alpha^k \subset \Sigma_\alpha$. And it is obvious that $\Sigma_\alpha \subset \bigcup_{k=1}^{\infty} \Sigma_\alpha^k$. Then we have $\bigcup_{k=1}^{\infty} \Sigma_\alpha^k = \Sigma_\alpha$.

Let $\varphi_k(x) = \sigma_{\Sigma_\alpha^k}(x)$, $\varphi(x) = \sigma_{\Sigma_\alpha}(x)$, $x \in S^{n-1}$, then $\varphi_k(x)$, $\varphi(x)$ are continuous with respect to x, and $\lim_{k\to\infty} \varphi_k(x) = \varphi(x)$. For each fixed x, $\varphi_k(x)$ is monotonously increasing, then by Dini Theorem, $\lim_{k\to\infty} \varphi_k(x) = \varphi(x)$ uniformly with respect to $x \in S^{n-1}$.

Therefore,

$$H(\Sigma_\alpha^k, \Sigma_\alpha) = \sup_{x\in S^{n-1}} \{|\sigma_{\Sigma_\alpha^k}(x) - \sigma_{\Sigma_\alpha}(x)| :\ x \in S^{n-1}\} \to 0,\ k \to \infty$$

uniformly for $\alpha \in [0, 1]$.

Then $\lim_{k\to\infty} D(v_k, v) = \lim_{k\to\infty} \sup_{0\le\alpha\le1} H(\Sigma_\alpha^k, \Sigma_\alpha) = 0$, uniformly for $t \in I$. □

Remark 4.1. From Theorem 4.1 and 4.2, if the forcing functions satisfies $\lim_{k\to\infty} D(f_k, f) = 0$, then the periodic solutions have the property: $\lim_{k\to\infty} D(v_k, v) = 0$. And if added the condition that f_k is monotone, then $\lim_{k\to\infty} D(v_k, v) = 0$ uniformly for $t \in I$.

Example 4.1. For (4.1), take $I = [0, 1]$, $f_k(t,\xi) = \mathrm{e}^t \otimes u_k + \xi$, $u_k \in \mathbf{E}^1$ and $[u_k]^\alpha = [1 + \frac{1}{k} + \alpha,\ 3 + \frac{1}{k} - \alpha]$, $\alpha \in [0, 1]$. For (3.1), take f as Example 3.1. We have f, f_k satisfies (i), (ii) in Theorem 3.2. And for condition (iii), there exists $G = 2.5 > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) = 0.4 > 0$, $m(\xi_0) = 0.21 > 0$ such that

$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$
$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f_k(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$

for a.e. $t \in I$. Then f, f_k satisfies the conditions in Theorem 4.1. So there exists the unique periodic solution $v_k : [0, 1] \to \mathbf{D}^1$ to (4.1) with $v_k(0) = v_k(1)$, where $v_k(t) = \mathrm{e}^t(t + \frac{\mathrm{e}}{1-\mathrm{e}}) \otimes u_k$, with $v_k(0) = v_k(1) = \frac{\mathrm{e}}{1-\mathrm{e}} \otimes u_k$. And $\lim_{k\to\infty} D(v_k, v) = 0$.

Example 4.2. For (4.1), take $I = [0, 1]$, $f_k(t,\xi) = \mathrm{e}^{-t} \otimes u_k + 2\xi$, $u_k \in \mathbf{E}^1$ and $[u_k]^\alpha = [\frac{1}{k} + \alpha,\ 2 - \frac{1}{k} - \alpha]$, $\alpha \in [0, 1]$. For (3.1), take $I = [0, 1]$, $f(t,\xi) = \mathrm{e}^{-t} \otimes u + 2\xi$, $u \in \mathbf{E}^1$ and $[u]^\alpha = [\alpha,\ 2 - \alpha]$, $\alpha \in [0, 1]$. We have f, f_k satisfies (i), (ii) in Theorem 3.2. And for condition (iii), there exists $G = \frac{2}{\mathrm{e}} > 0$ such that whenever $||\xi_0||_2 > G$ there exist $\delta(\xi_0) = \frac{1}{\mathrm{e}} > 0$, $m(\xi_0) = \frac{1}{\mathrm{e}^2} > 0$ such that

$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$
$$\inf\{\langle \xi, \zeta\rangle |\ \zeta \in [f_k(t,\xi)]^0,\ ||\xi - \xi_0||_2 < \delta(\xi_0)\} \ge m(\xi_0),$$

for a.e. $t \in I$. By definitions of f, f_k, we have

$$F_1(t, \xi(t), \alpha) \subset \cdots \subset F_k(t, \xi, \alpha) \subset \cdots \subset F(t, \xi, \alpha),$$

then f, f_k satisfies the conditions in Theorem 4.2. So there exists the unique periodic solution $v_k : [0, 1] \to \mathbf{D}^1$ to (4.1) with $v_k(0) = v_k(1)$, where $v_k(t) = -\frac{1}{3}\mathrm{e}^{2t}(\mathrm{e}^{-3t} + \frac{\mathrm{e}^{-1}}{\mathrm{e}+1}) \otimes u_k$, with $v_k(0) = v_k(1) = -\frac{1}{3}(1 + \frac{\mathrm{e}^{-1}}{\mathrm{e}+1}) \otimes u_k$. And $\lim_{k\to\infty} D(v_k, v) = 0$ uniformly for $t \in I$.

5. Conclusion and future expectations

In this paper, the periodic problems of first-order FDEs has been studied by using the differential inclusion method. Because of the H-derivative's limitations, this periodic problems could not be solved. But by means of differential inclusion method, the existence and uniqueness of periodic solutions for first-order FDEs have been obtained. And the structural stability of periodic solutions has also been discussed and established. The periodic solutions could maintain structure stability if the forcing functions has certain perturbations. And if the perturbation forcing functions are also monotone, then periodic solutions are uniformly convergence. That is to say, without the monotonicity, the periodic solutions could maintain structure stability without uniformity.

We considered the structural stability of periodic solutions for first-order FDEs in this paper. Because in the fuzzy number space $\mathbf{E}^n$, there is no Lyapunov function which exists in the sense of ordinary differential equation. How to find another way to replace the Lyapunov function is also interesting to discuss the stability of solutions for initial value problems of FDEs. And the bifurcation behaviors of solutions for initial value problems could be studied by the differential inclusion method in the future work.

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