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On the structural stability for two-point boundary value problems of undamped fuzzy differential equations ☆

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Abstract

In this paper, the structural stability for two-point boundary value problems of second order fuzzy differential equations (FDEs) has been studied by using differential inclusion method. In the sense of differential inclusion, this FDE is understood as a two-point boundary value problem of uncertain dynamical system for which exists a unique big solution and a unique solution. When the forcing function or boundary conditions have specific perturbations, the structural stability of big solutions is discussed by means of Green function. After that, by using tools of support function, the Dini Theorem and the Convergence Theorem in the differential inclusion theory, the structural stability of solutions has been discussed and established too.

Keywords: Fuzzy differential equation; Differential inclusion; Two-point boundary value problems; Structural stability; Green function

1. Introduction

To deal with the uncertainty or subjective information in the real world, fuzzy differential equations (FDEs) are widely used to build mathematical models. But solving the FDEs is quite different to general differential equations for its special subtraction. By Zadeh's Extension Principle, the subtraction of two fuzzy numbers $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is $u \ominus v = (u_1 - v_2,\ u_2 - v_1)$, where $u = (u_1, u_2)$ is the parametric representation of u (see [9]). On the other hand, the addition operation by Zadeh's Extension Principle is $u \oplus v = (u_1 + v_1,\ u_2 + v_2)$. So in the sense of Zadeh's Extension Principle, the derivative in FDEs is different to ordinary differential equations. In addition to the method of Zadeh's Extension Principle, H-derivatives and Bede's generalized derivatives are also effective methods to handle this problem by defining different derivatives. These approaches achieve the solutions to first-order FDEs (see [6,8,24]), second-order FDEs (see [3,5,12]), fractional FDEs (see [2]) and other FDEs (see [21]). Among above

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FDEs, the two-point boundary value problem of second-order FDEs is a classic problem. Recently, Sánchez et al. [29] studied this problem by using the sup-J extension principle, which is a generalization of the Zadeh's Extension Principle. They discussed the solution to linear fuzzy boundary value problem with interactive fuzzy boundary value. For nonlinear two-point boundary value problem as follows,

$$\begin{cases} x''(t) = f(t, x'(t), x(t)), \\ x(a) = A, \ x(b) = B, \end{cases} \tag{1.1}$$

where $I \stackrel{\triangle}{=} [a, b]$, $f : I \times \mathbf{E}_c \times \mathbf{E}_c \to \mathbf{E}_c$, A, $B \in \mathbf{E}_c$, where $\mathbf{E}_c$ is continuous fuzzy number space. In [12], M. Chen et al. found the conditions to make this problem having solution under H-derivatives. Diamond [16] pointed out the fact that support sets of the fuzzy solutions are nondecreasing in the sense of H-derivatives for first-order fuzzy differential equations. This conclusion is also valid for second-order fuzzy differential equations by [12], i.e., (1.1) hasn't solution under H-derivatives when the boundary condition is $A = B$ or $A > B$ ($A > B$ means α-level sets $[A]^\alpha$ of A and $[B]^\alpha$ of B satisfy $[B]^\alpha \subset [A]^\alpha$). Bede's generalized derivatives can handle this defect of H-derivatives by using switching points (see [5,24]). R. Rodríguez-López et al. [28] also investigated a kind of periodic boundary value problem of impulsive fuzzy differential equations. Besides, Hüllermeier [22], Diamond et al. [17,19] proposed that using differential inclusion could solve FDEs very well. The method of differential inclusion can also discuss the periodicity, stability and bifurcation behavior of FDEs which are difficult to study by other methods (see [16]). After differential inclusion method proposed, there are a lot of achievements by using this approach (see [1,7,8,10,11,13–15,23,25,30]). In [26], Li et al. solve the above two-point boundary value problem of second-order FDEs (1.1) by using differential inclusions. For the undamped situation of two-point boundary value problem, [9,26] studied the existence and uniqueness of solution by consider the following undamped two-point boundary value problem,

$$\begin{cases} x''(t) = f(t, x(t)), \\ x(a) = A, \ x(b) = B, \end{cases} \tag{1.2}$$

where $I = [a, b]$, $f : I \times \mathbf{E}_c \to \mathbf{E}_c$, A, $B \in \mathbf{E}_c$, as fuzzy differential inclusion problems:

$$\begin{cases} \xi''(t) \in f(t, \xi(t)), \\ \xi(a) \in A, \ \xi(b) \in B, \end{cases} \tag{1.3}$$

where $I = [a, b]$, $f : I \times \mathbb{R} \to \mathbf{E}_c$, $A, B \in \mathbf{E}_c$, and for $h \in \mathbb{R}$, $u \in \mathbf{E}_c$, $h \in u$ means $u(h) = \mu_u(h) > 0$, where μ_u is the membership function of u. Based on the results of [9,26], this paper uses the representation of fuzzy number and other tools of support function, Green function, Dini theorem to study the structural stability of the solution and the big solution when (1.3) has some specific perturbations. The big solution of (1.3) is the solution of the corresponding integral equation after extending f. For the damped situation of two-point boundary value problem, [12,15,26] used different strategies from the undamped situation to prove the existence of solutions to (1.1). Therefore, techniques of this paper can not be directly applied to discuss the structural stability of (1.1).

This paper is organized as follows. Section 2 provides the basic concepts. Section 3 recalls the definitions and theorem of the solution and the big solution to (1.3). In Section 4, we obtained the structural stability of big solutions for this problem. In Section 5, the structural stability of solutions has been proved. In Section 6, the conclusion is given.

2. Preliminaries

Like [9,26], we present basic concepts that are used in this study.

Definition 2.1. [18] Let $\mathbf{D}^1$ be the set of upper semicontinuous normal fuzzy sets with compact supports in $\mathbb{R}$ and $\mathbf{E}^1$ be the set of fuzzy convex subsets of $\mathbf{D}^1$.

Lemma 2.1 *(Stacking Theorem). [19] Let* $\{A_\alpha \subset \mathbb{R} |\ 0 \le \alpha \le 1\}$ *be a class of nonempty compact sets satisfying*

(i) $A_\beta \subset A_\alpha$ $(0 \le \alpha \le \beta \le 1)$,

(ii) $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ *for any nondecreasing sequence* $\{\alpha_n\}$ *in* $[0, 1]$ *satisfying* $\alpha_n \to \alpha$.

Then there exists $v \in \mathbf{D}^1$ such that $[v]^\alpha = A_\alpha (0 < \alpha \le 1)$. Especially if A_α is convex, $v \in \mathbf{E}^1$. On the other hand, if $v \in \mathbf{D}^1$, the α-level set $[v]^\alpha$ satisfies (i) and (ii) above. If $v \in \mathbf{E}^1$, $[v]^\alpha$ is convex.

The definitions and properties of $\mathbf{E}_c$ are given as follows.

Definition 2.2. [9] Let $\mathbf{E}_c = \{u \in \mathbf{E}^1 | \, u_1(\alpha) = \min[u]^\alpha, \; u_2(\alpha) = \max[u]^\alpha$ be continuous on $[0, 1]\}$, i.e. $u \in \mathbf{E}_c$ satisfies the following conditions (i)-(v):

(i) u is normal, i.e., $\exists m \in \mathbb{R}$ such that $u(m) = 1$,
(ii) $[u]^0 = cl\{h \in \mathbb{R} | u(h) > 0\}$ is bounded in $\mathbb{R}$,
(iii) u is fuzzy convex in $\mathbb{R}$,
(iv) u is upper semicontinuous on $\mathbb{R}$,
(v) Denote $[u]^\alpha = \{h \in \mathbb{R} | u(h) \ge \alpha\}$ $(0 < \alpha \le 1)$, $u_1(\alpha) = \min[u]^\alpha$, $u_2(\alpha) = \max[u]^\alpha$ $(\alpha \in [0, 1])$, then $u_1(\alpha), u_2(\alpha)$ are continuous on $[0, 1]$.

We call $u \in \mathbf{E}_c$ continuous fuzzy number and fuzzy number in abbreviation.

Theorem 2.1. *[9] For $u \in \mathbf{E}_c$, the following (1)-(3) hold:*

(1) $u_1(\alpha), u_2(\alpha)$ are continuous on $[0, 1]$,
(2) $u_1(\alpha)$ is monotone increasing and $u_2(\alpha)$ is monotone decreasing,
(3) $u_1(1) \le u_2(1)$.
Conversely, if $i(\alpha), \; s(\alpha) : [0, 1] \to \mathbb{R}$ satisfy (1)-(3) above, denote

$$u(h) = \begin{cases} \sup\{\alpha \in [0, 1] | \, i(\alpha) \le h \le s(\alpha)\}, & h \in [i(0), s(0)]; \\ 0, & h \notin [i(0), s(0)]. \end{cases}$$

Then $\exists u \in \mathbf{E}_c$ such that $[u]^\alpha = [i(\alpha), s(\alpha)]$, $u_1(\alpha) = i(\alpha)$, $u_2(\alpha) = s(\alpha)$, $\alpha \in [0, 1]$.

From Definition 2.2 and Theorem 2.1, $u_1(\alpha)$ and $u_2(\alpha)$ can be used to represent $u \in \mathbf{E}_c$, i.e., $u = (u_1(\alpha), u_2(\alpha))$, $\alpha \in [0, 1]$, or $u = (u_1, u_2)$ for simplicity.

Remark 2.1. For $x \in \mathbb{R}$, it can be considered as a special point in $\mathbf{E}_c$ and represented as $x = (x, x)$, $\forall \alpha \in [0, 1]$.

Theorem 2.2. *[9] $\mathbf{E}_c$ is a closed convex cone in Banach space $X \overset{\triangle}{=} C[0, 1] \times C[0, 1]$, and then it is a complete metric space.*

Let $D : \mathbf{D}^1 \times \mathbf{D}^1$ be the metric given by $D(u, v) = \sup\limits_{0 \le \alpha \le 1} H([u]^\alpha, [v]^\alpha)$, where H is the Hausdorff metric defined on the family $P_k(\mathbb{R})$ of all compact subsets of $\mathbb{R}$. It is well-known (see [18,31]) that if A and B are elements in the family $P_{kc}(\mathbb{R})$ of all convex and compact subsets of $\mathbb{R}$, then $H(A, B)$ can be characterized by

$$H(A, B) = \sup\{|\sigma_A(x) - \sigma_B(x)| : \; x = \pm 1\},$$

where $\sigma_A(x) = \sup\{\langle x, y\rangle : \; y \in A\}$, and $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}$. And the convexity of A and B does not affect the validity of this characterization. So $H(A, B) = \sup\{|\sigma_A(x) - \sigma_B(x)| : \; x = \pm 1\}$ is also valid for any $A, \; B \in P_k(\mathbb{R})$. The zero in $\mathbf{E}_c$ is the function $\hat{0} : \; \mathbb{R} \to [0, 1]$ defined by

$$\hat{0}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let $|| \cdot ||_p$ be Puri's norm on $\mathbf{E}_c$ (see [27]), then we have $||u|| = ||u||_p = D(u, \hat{0})$, $u \in \mathbf{E}_c$.

Denote:
$C(I, X) = \{x | x : I \to X$ is continuous on I, $I = [a, b] \subset \mathbb{R}\}$,
$C(I, \mathbf{E}_c) = \{x | x : I \to \mathbf{E}_c$ is continuous on I, $I = [a, b] \subset \mathbb{R}\}$.

For $C(I, X)$, we introduce the norm $||x||_\infty = \sup\limits_{t \in I} ||x(t)||$, $\forall x \in C(I, X)$, then $C(I, X)$ is a Banach space. $C(I, \mathbf{E}_c)$ is a closed convex cone in $C(I, X)$ (see [13]).

For $u, v \in \mathbf{D}^1$, $u \subset v$ if and only if $[u]^\alpha \subset [v]^\alpha$ ($\forall \alpha \in [0,1]$). For $\{u_n\} \subset \mathbf{D}^1$ is monotone, if $u_{n+1} \subset u_n$ $(n = 1, 2, ...)$ or $u_n \subset u_{n+1}$ $(n = 1, 2, ...)$ (see [31]).

On the fuzzy space $\mathbf{E}_c$, the following properties are used in this paper.

Definition 2.3. [9] Let $f : I \to \mathbf{E}_c$ (or X), $t_0 \in I$. If $\forall \varepsilon > 0$, $\exists \delta > 0$, such that $||f(t) - f(t_0)|| < \varepsilon$ whenever $t \in I$ and $|t - t_0| < \delta$, then we say that f is continuous at t_0. If f is continuous at each point of I, we say that f is continuous on I.

Definition 2.4. [9] Let $f : I \to \mathbf{E}_c$ (or X), $J \in \mathbf{E}_c$ (or X). If for any partition $\triangle$ of I:

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b,$$

and $\forall \tau_k \in [t_{k-1}, t_k]$ $(k = 1, 2, \cdots, n)$, we have $\lim\limits_{\lambda(\triangle)\to 0} \sum\limits_{k=1}^{n} f(\tau_k) \triangle t_k = J$, where $\lambda(\triangle) = \max\limits_{1\le k\le n} \{\triangle t_k\}$, $\triangle t_k = t_k - t_{k-1}$ $(k = 1, 2, \cdots, n)$, then we say that f is integrable on I and denote $J = \int_a^b f(t)dt$.

Proposition 2.1. *[9] Let $f : I \to \mathbf{E}_c$ be continuous on I, then f is integrable on I and*

$$\left\| \int_a^b f(t)dt \right\| \le \int_a^b ||f(t)||dt.$$

Lemma 2.2 *(Dini Theorem). [20] If a monotone sequence of continuous real-valued functions $\{f_n\}$ $(n = 1, 2, \cdots)$ converges to a continuous function f on a compact set K, then the $\{f_n\}$ converges uniformly to f on K.*

Lemma 2.3. *[17] Let Ω be an open set of $\mathbb{R} \times \mathbb{R}$, $f : \Omega \to \mathbf{E}_c$ is upper semicontinuous, and let $L(\cdot, \cdot, \alpha) = [f(\cdot, \cdot)]^\alpha : \Omega \to P_{kc}(\mathbb{R})$ for each $\alpha \in [0, 1]$. Then $L(\cdot, \cdot, \alpha)$ is upper semicontinuous on Ω.*

Lemma 2.4 *(Convergence Theorem). [4] Let $\mathcal{F}$ be a proper semicontinuous map from a Hausdorff locally convex space $\mathbf{X}$ to a closed convex subset of a Banach space $\mathbf{Y}$. Let x_k and y_k be measurable functions from T to $\mathbf{X}$ and $\mathbf{Y}$, respectively, where T is an interval of $\mathbb{R}$. If*

i) $x_k(\cdot)$ converges almost everywhere to a function $x(\cdot)$ from T to $\mathbf{X}$,

ii) $y_k(\cdot)$ belongs to $L^1(I, \mathbf{Y})$ and converges weakly to $y(\cdot)$ in $L^1(I, \mathbf{Y})$,

iii) there exists $k_0 = k_0(t, N)$ such that $\forall k \ge k_0$, $(x_k(t), y_k(t)) \in Gr(\mathcal{F}) + N$ for almost all $t \in T$ and for every neighborhood N of 0 in $\mathbf{X} \times \mathbf{Y}$, where $Gr(\mathcal{F})$ is the graph of $\mathcal{F}$,

then $(x(t), y(t)) \in Gr(\mathcal{F})$ for almost all $t \in T$, i.e., $y(t) \in \mathcal{F}(x(t))$ for almost all $t \in T$.

3. The existence of solutions

In this section, we present the solution and big solution for two-point boundary value problem of undamped uncertain dynamical system (1.3).

Like the technique of solving FDEs in [9,26], the Green function can be used to deal with it. Consider the Green function

$$G(t,s) = \begin{cases} \frac{(b-t)(a-s)}{b-a}, & a \le s \le t \le b, \\ \frac{(b-s)(a-t)}{b-a}, & a \le t \le s \le b, \end{cases}$$

and $w(t) = \frac{A(b-t)+B(t-a)}{b-a}$, then $w(a) = A$, $w(b) = B$ and $\int_a^b |G(t,s)|ds \le \frac{(b-a)^2}{8}$.

By taking the α-level set of (1.3), the following class of differential inclusions are taken into consideration.

$$\xi''(t) \in [f(t, \xi(t))]^\alpha,\ \xi(a) \in [A]^\alpha,\ \xi(b) \in [B]^\alpha\ (\alpha \in [0, 1]). \tag{3.1}$$

If $f \in C(I \times \mathbb{R}, \mathbf{E}_c)$, i.e., $f(t, \xi(t))$ is continuous on $I \times \mathbb{R}$, from (3.1) we have

$$\int_a^b G(t,s)\xi''(s)ds \in \int_a^b G(t,s)[f(s,\xi(s))]^\alpha ds,$$

$$\frac{\xi(a)(b-t)+\xi(b)(t-a)}{b-a} \in [w(t)]^\alpha,$$

and furthermore

$$\xi(t) \in [w(t)]^\alpha + \int_a^b G(t,s)[f(s,\xi(s))]^\alpha ds \ (0 \le \alpha \le 1).$$

In order to study the scope of trajectories of solutions to (1.3), we need to extend $[f(t,\xi(t))]^\alpha : I \times \mathbb{R} \to P_{kc}(\mathbb{R})$ to $F_\alpha(t,x) : I \times \mathbf{E}_c \to P_{kc}(\mathbb{R})$, where $P_{kc}(\mathbb{R})$ is the set of compact convex subsets of $\mathbb{R}$. For $x = (x_1, x_2) \in \mathbf{E}_c$, we define:

$$F_\alpha(t,x) = \overline{co}\left(\bigcup_{\xi(t)\in[x]^\alpha} [f(t,\xi(t))]^\alpha\right) \ (0 \le \alpha \le 1),$$

where $\overline{co}(A)$ is the closed convex hull of set A.

Lemma 3.1. *[26] Let $f \in C(I \times \mathbb{R}, \mathbf{E}_c)$ and $F_\alpha(t,x)$ be extended from f by above definition. Then there exists $F : I \times \mathbf{E}_c \to \mathbf{E}_c$ such that*

$$[F(t,x)]^\alpha = F_\alpha(t,x) \ (0 \le \alpha \le 1), \ t \in I, \ x \in \mathbf{E}_c.$$

Remark 3.1. From the definition of $F_\alpha(t,x)$ and Lemma 3.1, for $x \in \mathbb{R}$, $F(t,x) = f(t,x)$. This also shows that the above extension is reasonable.

Instead of studying (1.3), we study the following integral equation:

$$x(t) = w(t) + \int_a^b G(t,s) \otimes F(s,x(s))ds, \tag{3.2}$$

where $x : I \to \mathbf{E}_c$, $\otimes$ is the scalar multiplication of Zadeh's Extension Principle.

By [26], the definitions of solutions to (1.3) and (3.1) are given as follows.

Definition 3.1. If $\xi'(t)$ is absolutely continuous, $\xi(a) \in [A]^\alpha$, $\xi(b) \in [B]^\alpha$, and $\xi''(t) \in [f(t,\xi(t))]^\alpha$ a.e. on I, then we call $\xi(t)$ a solution of (3.1) $(0 \le \alpha \le 1)$ and $\Sigma_\alpha(A,B;t) = \{\xi(t)|\xi(t)$ is a solution of (3.1). $\}$ $(0 \le \alpha \le 1)$ the set of solutions of (3.1). If there exists $v : I \to \mathbf{D}^1$ such that $[v(t)]^\alpha = \Sigma_\alpha(A,B;t)$ $(0 \le \alpha \le 1)$, then we call $v(t)$ a solution of (1.3).

Definition 3.2. The fuzzy number value function $x : I \to \mathbf{E}_c$ satisfying (3.2) is called a big solution of (1.3).

From above definitions, the solution $v : I \to \mathbf{D}^1$ of (1.3) is defined by solution sets $\Sigma_\alpha(A,B;t)$ of (3.1) if $[v(t)]^\alpha = \Sigma_\alpha(A,B;t)$ $(0 \le \alpha \le 1)$. On the other hand, the big solution of (1.3) is the solution of (3.2). Next the relationship between the solution and the big solution will be discussed. Before that, properties of $F : I \times \mathbf{E}_c \to \mathbf{E}_c$ will be introduced first.

Lemma 3.2. *[26] If $f : I \times \mathbb{R} \to \mathbf{E}_c$ satisfies:*

(i) $f(t,\eta)$ is continuous on $I \times \mathbb{R}$,

(ii) There exists a Lebesgue integrable function $p : I \to \mathbb{R}^+$ such that

$$||f(t,\eta) - f(t,\zeta)|| \le p(t)|\eta - \zeta|,$$

$\forall\eta,\zeta\in\mathbb{R}$ and $t\in I$, then $F: I\times\mathbf{E}_c\to\mathbf{E}_c$ satisfies:
(1) $F(t,x)$ is continuous on $I\times\mathbf{E}_c$,
(2) $\forall x,y\in\mathbf{E}_c$, we have

$$||F(t,x)-F(t,y)||\le p(t)||x-y||,\ (t\in I).$$

Theorem 3.1. *[26] Suppose that $f: I\times\mathbb{R}\to\mathbf{E}_c$ satisfies:*
(i) $f(t,\eta)$ is continuous on $I\times\mathbb{R}$,
(ii) There exists a Lebesgue integrable function $p: I\to\mathbb{R}^+$ such that

$$||f(t,\eta)-f(t,\zeta)||\le p(t)|\eta-\zeta|,$$

for $\forall\eta,\zeta\in\mathbb{R}$ and $t\in I$.
(iii) $k=\sup\limits_{t\in I}\int_a^b |G(t,s)|p(s)ds<1$.
Then there exists a unique solution of (1.3) $v: I\to\mathbf{D}^1$ such that $v(t)\subset x^*(t)$ $(t\in I)$, where $x^*: I\to\mathbf{E}_c$ is the unique big solution of (1.3).

Remark 3.2. From Theorem 3.1, the existence and uniqueness of the solution and the big solution of (1.3) could be established under some conditions, and the solution is contained in the corresponding big solution. By the Definition 3.2, the big solution of (1.3) actually the solution of integral equation (3.2). However, the introduction of the concept of big solution plays a crucial role in proving the existence of the solution to (1.3) by verifying the boundedness of the set of solutions to (3.1) (see [26]).

4. The structural stability of big solution

In section 3, the existence of big solution for two-point boundary value problem has been introduced. The big solution is the solution to integral equation (3.2). The structural stability of integral equation is also an interesting problem that we are concerned about. Next we will discuss structural stability to the integral equation (3.2) if given some perturbations.

4.1. Perturbations in the forcing function

Considering a certain perturbation to the forcing function as follows:

$$\begin{cases}\xi''(t)\in f_n(t,\xi(t)),\\ \xi(a)\in A,\ \xi(b)\in B,\end{cases}\tag{4.1}$$

where $I=[a,b]$, $f_n\in C(I\times\mathbb{R},\mathbf{E}_c)$, $A,B\in\mathbf{E}_c$. By taking the α-level set of (4.1), the following class of differential inclusions are taken into consideration.

$$\xi''(t)\in[f_n(t,\xi(t))]^\alpha,\ \xi(a)\in[A]^\alpha,\ \xi(b)\in[B]^\alpha\ (\alpha\in[0,1]).$$

Theorem 4.1. *Suppose that $f,\ f_n: I\times\mathbb{R}\to\mathbf{E}_c$ satisfies (i), (ii) and (iii) in Theorem 3.1, and (iv) $\{f_n(t,\xi(t))\}$ is monotone with respect to n and $\lim\limits_{n\to\infty}D(f_n(t,\xi(t)),f(t,\xi(t))=0$ for each $t\in I$.*

Then the big solution $x_n^*: I\to\mathbf{E}_c$ to (4.1) and the big solution $x^*: I\to\mathbf{D}^1$ to (1.3) satisfy $\lim\limits_{n\to\infty}D(x_n^*(t),x^*(t))=0$ uniformly with respect to $t\in I$.

Proof. By the Lemma 3.1 and Theorem 3.1, there exist big solutions $x_n^*, x^*: I\to\mathbf{E}_c$, and

$$x_n^*(t)=w(t)+\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds\ (t\in I),$$

$$x^*(t) = w(t) + \int_a^b G(t,s) \otimes F(s, x^*(s))ds \ (t \in I),$$

where $F_n(t, x(t))$ and $F(t, x(t))$ are extended from $f_n(t, \xi(t))$ and $f(t, \xi(t))$, respectively. As $G(t,s)$ never changes the sign on I and $G(t,s) < 0$. By Definition 2.3 and Proposition 2.1, we have

$$\begin{aligned}
D(x_n^*(t), x^*(t)) &= D(w(t) + \int_a^b G(t,s) \otimes F_n(s, x_n^*(s))ds, w(t) + \int_a^b G(t,s) \otimes F(s, x^*(s))ds) \\
&= D(\int_a^b G(t,s) \otimes F_n(s, x_n^*(s))ds, \int_a^b G(t,s) \otimes F(s, x^*(s))ds) \\
&\le \int_a^b D(G(t,s) \otimes F_n(s, x_n^*(s)), G(t,s) \otimes F(s, x^*(s)))ds \\
&\le \int_a^b |G(t,s)| \cdot D(F_n(s, x_n^*(s)), F(s, x^*(s)))ds \\
&\le \int_a^b |G(t,s)| \cdot [D(F_n(s, x_n^*(s)), F_n(s, x^*(s))) + D(F_n(s, x^*(s)), F(s, x^*(s)))]ds.
\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}
&\int_a^b |G(t,s)| \cdot [D(F_n(s, x_n^*(s)), F_n(s, x^*(s))) + D(F_n(s, x^*(s)), F(s, x^*(s)))]ds \\
&\qquad \le \int_a^b |G(t,s)| \cdot [p(s)D(x_n^*(s), x^*(s)) + D(F_n(s, x^*(s)), F(s, x^*(s)))]ds.
\end{aligned}$$

By the definition of $||\cdot||_\infty$ and the condition (iii) of Theorem 3.1, we have

$$\begin{aligned}
&\int_a^b |G(t,s)| \cdot [p(s)D(x_n^*(s), x^*(s)) + D(F_n(s, x^*(s)), F(s, x^*(s)))]ds \\
&\qquad \le \int_a^b |G(t,s)|p(s)ds||x_n^* - x^*||_\infty + \int_a^b |G(t,s)|ds||F_n - F||_\infty \\
&\le k||x_n^* - x^*||_\infty + \frac{(b-a)^2}{8}||F_n - F||_\infty.
\end{aligned}$$

Therefore,

$$D(x_n^*(t), x^*(t)) \le \sup_{t \in I} D(x_n^*(t), x^*(t)) = ||x_n^* - x^*||_\infty \le \frac{(b-a)^2}{8(1-k)}||F_n - F||_\infty.$$

As $\{f_n(t, \xi(t))\}$ is monotone with respect to n and $\lim_{n\to\infty} D(f_n(t, \xi(t)), f(t, \xi(t)) = 0$ for each $t \in I$, then $\{[f_n(t, \xi(t))]^\alpha\}$ is monotone with respect to n and

$$\lim_{n\to\infty} \sup_{\alpha \in I} H([f_n(t, \xi(t))]^\alpha, [f(t, \xi(t))]^\alpha) = 0,$$

for each $t \in I$.

For each $t \in I$, let $\varphi_n(t,\xi,q,\alpha) = \sigma_{[f_n(t,\xi(t))]^\alpha}(q)$, $\varphi(t,\xi,q,\alpha) = \sigma_{[f(t,\xi(t))]^\alpha}(q)$, where $q = \pm 1$, then $\varphi_n(t,\xi,q,\alpha)$, $\varphi(t,\xi,q,\alpha)$ are continuous with respect to (t,ξ,q,α), and $\lim_{n\to\infty}\varphi_n(t,\xi,q,\alpha) = \varphi(t,\xi,q,\alpha)$ uniformly with respect to $\alpha \in [0,1]$ for each $(t,\xi,q) \in I \times [x]^\alpha \times \{\pm 1\}$. For each fixed (t,ξ,q), $\{\varphi_n(t,\xi,q,\alpha)\}$ is monotonously decreasing or increasing with respect to n. Then $\forall x \in \mathbf{E}_c$, $[x]^\alpha \in P_{kc}(\mathbb{R})$, by Dini Theorem, $\lim_{n\to\infty}\varphi_n(t,\xi,q,\alpha) = \varphi(t,\xi,q,\alpha)$ uniformly with respect to $(t,\xi,q,\alpha) \in I \times [x]^\alpha \times \{\pm 1\} \times [0,1]$.

By the definition of $F_n(t,x)$, we have

$$[F_n(t,x)]^\alpha = \overline{co}\left(\bigcup_{\xi\in[x]^\alpha}[f_n(t,\xi(t))]^\alpha\right) \quad (0 \le \alpha \le 1),$$

then $\lim_{n\to\infty}\sigma_{[F_n(t,x)]^\alpha}(q) = \sigma_{[F(t,x)]^\alpha}(q)$ uniformly with respect to $(t,\xi,q,\alpha) \in I \times [x]^\alpha \times \{\pm 1\} \times [0,1]$. Furthermore,

$$\lim_{n\to\infty} H([F_n(t,x)]^\alpha, [F(t,x)]^\alpha) = 0,$$

uniformly with respect to $(t,\alpha) \in I \times [0,1]$. Therefore,

$$\lim_{n\to\infty} D(F_n(t,x^*(t)), F(t,x^*(t))) = \lim_{n\to\infty}\sup_{\alpha\in[0,1]} H([F_n(t,x^*(t))]^\alpha, [F(t,x^*(t))]^\alpha) = 0.$$

Then, we have

$$\lim_{n\to\infty}||F_n - F||_\infty = \lim_{n\to\infty}\sup_{t\in I} D(F_n(t,x^*(t)), F(t,x^*(t)) = 0.$$

Therefore,

$$\lim_{n\to\infty} D(x_n^*(t), x^*(t)) \le \lim_{n\to\infty}\frac{(b-a)^2}{8(1-k)}||F_n - F||_\infty = 0,$$

uniformly with respect to $t \in I$. $\square$

4.2. Perturbations on boundary conditions

Considering a certain perturbation to boundary conditions as follows:

$$\begin{cases} \xi''(t) \in f(t,\xi(t)), \\ \xi(a) \in A_n,\ \xi(b) \in B_n, \end{cases} \tag{4.2}$$

where $I = [a,b]$, $f \in C(I \times \mathbb{R}, \mathbf{E}_c)$, $A_n, B_n \in \mathbf{E}_c$. By taking the α-level set of (4.2), the following class of differential inclusions are taken into consideration.

$$\xi''(t) \in [f(t,\xi(t))]^\alpha,\ \xi(a) \in [A_n]^\alpha,\ \xi(b) \in [B_n]^\alpha\ (\alpha \in [0,1]).$$

Theorem 4.2. *Suppose that $f : I \times \mathbb{R} \to \mathbf{E}_c$ satisfies (i), (ii) and (iii) in Theorem 3.1, and (iv) $\lim_{n\to\infty} D(A_n, A) = 0$, $\lim_{n\to\infty} D(B_n, B) = 0$.*

Then the big solution $x_n^* : I \to \mathbf{E}_c$ to (4.2) and the big solution $x^* : I \to \mathbf{D}^1$ to (1.3) satisfy $\lim_{n\to\infty} D(x_n^*(t), x^*(t)) =$ 0 uniformly with respect to $t \in I$.

Proof. By the Lemma 3.1 and Theorem 3.1, there exist big solutions $x_n^*, x^* : I \to \mathbf{E}_c$, and

$$x_n^*(t) = w_n(t) + \int_a^b G(t,s) \otimes F(s, x_n^*(s))ds\ (t \in I),$$

$$x^*(t) = w(t) + \int_a^b G(t,s) \otimes F(s, x^*(s))ds\ (t \in I),$$

where $w_n(t) = \frac{A_n(b-t)+B_n(t-a)}{b-a}$.

By Definition 2.3 and Proposition 2.1, we have

$$\begin{aligned} D(x_n^*(t), x^*(t)) &= D(w_n(t) + \int_a^b G(t,s) \otimes F(s, x_n^*(s))ds, w(t) + \int_a^b G(t,s) \otimes F(s, x^*(s))ds) \\ &\le D(w_n(t) + \int_a^b G(t,s) \otimes F(s, x_n^*(s))ds, w(t) + \int_a^b G(t,s) \otimes F(s, x_n^*(s))ds) \\ &\quad + D(w(t) + \int_a^b G(t,s) \otimes F(s, x_n^*(s))ds, w(t) + \int_a^b G(t,s) \otimes F(s, x^*(s))ds) \\ &\le D(w_n(t), w(t)) + D(\int_a^b G(t,s) \otimes F(s, x_n^*(s))ds, \int_a^b G(t,s) \otimes F(s, x^*(s))ds). \end{aligned}$$

By the definition of $||\cdot||_\infty$ and Lemma 3.2, we have

$$\begin{aligned} &D(w_n(t), w(t)) + D(\int_a^b G(t,s) \otimes F(s, x_n^*(s))ds, \int_a^b G(t,s) \otimes F(s, x^*(s))ds) \\ &\qquad \le D(w_n(t), w(t)) + \int_a^b |G(t,s)|p(s)ds||x_n^* - x^*||_\infty. \end{aligned}$$

By the definition of $w_n(t)$, $w(t)$ and the condition (iii) of Theorem 3.1, we have

$$\begin{aligned} &D(w_n(t), w(t)) + \int_a^b |G(t,s)|p(s)ds||x_n^* - x^*||_\infty \\ &\le D(w_n(t), w(t)) + k||x_n^* - x^*||_\infty \\ &= D(\frac{A_n(b-t)+B_n(t-a)}{b-a}, \frac{A(b-t)+B(t-a)}{b-a}) + k||x_n^* - x^*||_\infty \\ &\le \frac{1}{b-a}[D(A_n(b-t)+B_n(t-a), A_n(b-t)+B(t-a)) \\ &\quad + D(A_n(b-t)+B(t-a), A(b-t)+B(t-a))] + k||x_n^* - x^*||_\infty \\ &\le \frac{1}{b-a}[D(B_n(t-a), B(t-a)) + D(A_n(b-t), A(b-t))] + k||x_n^* - x^*||_\infty \\ &\le \frac{t-a}{b-a}D(B_n, B) + \frac{b-t}{b-a}D(A_n, A) + k||x_n^* - x^*||_\infty. \end{aligned}$$

Furthermore,

$$\begin{aligned} D(x_n^*(t), x^*(t)) &\le \sup_{t\in I} D(x_n^*(t), x^*(t)) \\ &= ||x_n^* - x^*||_\infty \\ &\le \frac{1}{1-k}[\frac{t-a}{b-a}D(B_n, B) + \frac{b-t}{b-a}D(A_n, A)]. \end{aligned}$$

As $\lim_{n\to\infty} D(A_n, A) = 0$, $\lim_{n\to\infty} D(B_n, B) = 0$, then

$$\lim_{n\to\infty} D(x_n^*(t), x^*(t)) \le \lim_{n\to\infty} \frac{1}{1-k}[\frac{t-a}{b-a}D(B_n, B) + \frac{b-t}{b-a}D(A_n, A)]$$

$$
\begin{aligned}
&= \frac{1}{1-k}[\lim_{n\to\infty}\frac{t-a}{b-a}D(B_n,B)+\lim_{n\to\infty}\frac{b-t}{b-a}D(A_n,A)] \\
&\leq \frac{1}{1-k}[\lim_{n\to\infty}\frac{b-a}{b-a}D(B_n,B)+\lim_{n\to\infty}\frac{b-a}{b-a}D(A_n,A)] \\
&= \frac{1}{1-k}[\lim_{n\to\infty}D(B_n,B)+\lim_{n\to\infty}D(A_n,A)] \\
&= 0,
\end{aligned}
$$

uniformly with respect to $t \in I$. □

4.3. Perturbations on the forcing function and boundary conditions

Considering small perturbations to the forcing function and boundary conditions as follows:

$$
\begin{cases}
\xi''(t) \in f_n(t,\xi(t)), \\
\xi(a) \in A_n,\ \xi(b) \in B_n,
\end{cases}
\tag{4.3}
$$

where $I=[a,b]$, $f_n \in C(I\times\mathbb{R},\mathbf{E}_c)$, $A_n, B_n \in \mathbf{E}_c$. By taking the α-level set of (4.3), the following class of differential inclusions are taken into consideration.

$$\xi''(t) \in [f_n(t,\xi(t))]^\alpha,\ \xi(a)\in[A_n]^\alpha,\ \xi(b)\in[B_n]^\alpha\ (\alpha\in[0,1]).$$

Theorem 4.3. *Suppose that $f,\ f_n : I\times\mathbb{R}\to\mathbf{E}_c$ satisfies (i), (ii) and (iii) in Theorem 3.1, and*

(iv) $\{f_n(t,\xi(t))\}$ is monotone with respect to n and $\lim_{n\to\infty}D(f_n(t,\xi(t)),f(t,\xi(t))=0$ for each $t\in I$,

(v) $\lim_{n\to\infty}D(A_n,A)=0,\ \lim_{n\to\infty}D(B_n,B)=0.$

Then the big solution $x_n^*: I\to\mathbf{E}_c$ to (4.3) and the big solution $x^*: I\to\mathbf{D}^1$ to (1.3) satisfy $\lim_{n\to\infty}D(x_n^*(t),x^*(t))=0$ uniformly with respect to $t\in I$.

Proof. By the Lemma 3.1 and Theorem 3.1, there exist big solutions $x_n^*, x^*: I\to\mathbf{E}_c$, and

$$x_n^*(t)=w_n(t)+\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds\ (t\in I),$$

$$x^*(t)=w(t)+\int_a^b G(t,s)\otimes F(s,x^*(s))ds\ (t\in I),$$

where $w_n(t)=\frac{A_n(b-t)+B_n(t-a)}{b-a}$, $w(t)=\frac{A(b-t)+B(t-a)}{b-a}$.

By Definition 2.3 and Proposition 2.1, we have

$$
\begin{aligned}
D(x_n^*(t),x^*(t)) &= D(w_n(t)+\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds,\ w(t)+\int_a^b G(t,s)\otimes F(s,x^*(s))ds) \\
&\leq D(w_n(t)+\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds,\ w_n(t)+\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds) \\
&\quad + D(w_n(t)+\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds,\ w(t)+\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds) \\
&\quad + D(w(t)+\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds,\ w(t)+\int_a^b G(t,s)\otimes F(s,x^*(s))ds)
\end{aligned}
$$

$$\leq D(\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds, \int_a^b G(t,s)\otimes F(s,x_n^*(s))ds)$$
$$+ D(w_n(t),w(t)) + D(\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds, \int_a^b G(t,s)\otimes F(s,x^*(s))ds).$$

By the definition of $||\cdot||_\infty$, Lemma 3.2 and the condition (iii) of Theorem 3.1, we have

$$D(\int_a^b G(t,s)\otimes F_n(s,x_n^*(s))ds, \int_a^b G(t,s)\otimes F(s,x_n^*(s))ds)$$
$$+ D(w_n(t),w(t)) + D(\int_a^b G(t,s)\otimes F(s,x_n^*(s))ds, \int_a^b G(t,s)\otimes F(s,x^*(s))ds)$$
$$\leq \int_a^b |G(t,s)|ds||F_n - F||_\infty + D(w_n(t),w(t)) + \int_a^b |G(t,s)|p(s)ds||x_n^* - x^*||_\infty$$
$$\leq \frac{(b-a)^2}{8}||F_n - F||_\infty + D(w_n(t),w(t)) + k||x_n^* - x^*||_\infty.$$

By the definition of $w_n(t)$, $w(t)$, we have

$$\frac{(b-a)^2}{8}||F_n - F||_\infty + D(w_n(t),w(t)) + k||x_n^* - x^*||_\infty$$
$$= \frac{(b-a)^2}{8}||F_n - F||_\infty + D(\frac{A_n(b-t)+B_n(t-a)}{b-a}, \frac{A(b-t)+B(t-a)}{b-a})$$
$$+ k||x_n^* - x^*||_\infty$$
$$\leq \frac{(b-a)^2}{8}||F_n - F||_\infty + \frac{1}{b-a}[D(A_n(b-t)+B_n(t-a), A_n(b-t)+B(t-a))$$
$$+ D(A_n(b-t)+B(t-a), A(b-t)+B(t-a))] + k||x_n^* - x^*||_\infty$$
$$\leq \frac{(b-a)^2}{8}||F_n - F||_\infty + \frac{1}{b-a}[D(B_n(t-a), B(t-a)) + D(A_n(b-t), A(b-t))]$$
$$+ k||x_n^* - x^*||_\infty$$
$$\leq \frac{(b-a)^2}{8}||F_n - F||_\infty + \frac{t-a}{b-a}D(B_n,B) + \frac{b-t}{b-a}D(A_n,A) + k||x_n^* - x^*||_\infty.$$

From (iv), (v) and the proofs of Theorem 4.1 and Theorem 4.2, we can conclude that

$$\lim_{n\to\infty} D(x_n^*(t),x^*(t)) \leq \lim_{n\to\infty}\frac{1}{1-k}(\frac{(b-a)^2}{8}||F_n - F||_\infty + D(B_n,B) + D(A_n,A))$$
$$= 0,$$

uniformly with respect to $t \in I$. □

Example 4.1. In (1.3) and (4.1), take $I = [0,1]$, $f(t,\xi(t)) = (\alpha, 4-\alpha)$, $\alpha \in [0,1]$, $f_n(t,\xi(t)) = (\alpha + \frac{1}{n}, 4-\alpha-\frac{1}{n})$, $\alpha \in [0,1]$, $A = B \in \mathbf{E}_c$. Then big solutions of (1.3) and (4.1) are $x^*(t) = A + \frac{t(1-t)}{2}[(-1)\otimes f]$ and $x_n^*(t) = A + \frac{t(1-t)}{2}[(-1)\otimes f_n]$. It can be concluded that $\lim_{n\to\infty} D(x_n^*(t),x^*(t)) = 0$ uniformly with respect to $t \in I$.

Let $A = B = (1+\alpha, 3-\alpha)$, $\alpha \in [0,1]$. Membership functions $\mu_{x^*}(h,t)$ and $\mu_{x_n^*}(h,t)$ $(n = 1,3,10)$ of solutions $x^*(t)$ and $x_n^*(t)$ $(n = 1,3,10)$ are shown in Fig. 1. The graph d in Fig. 1 shows the membership function $\mu_{x^*}(h,t)$ which is the case of $n = \infty$ by Theorem 4.1.

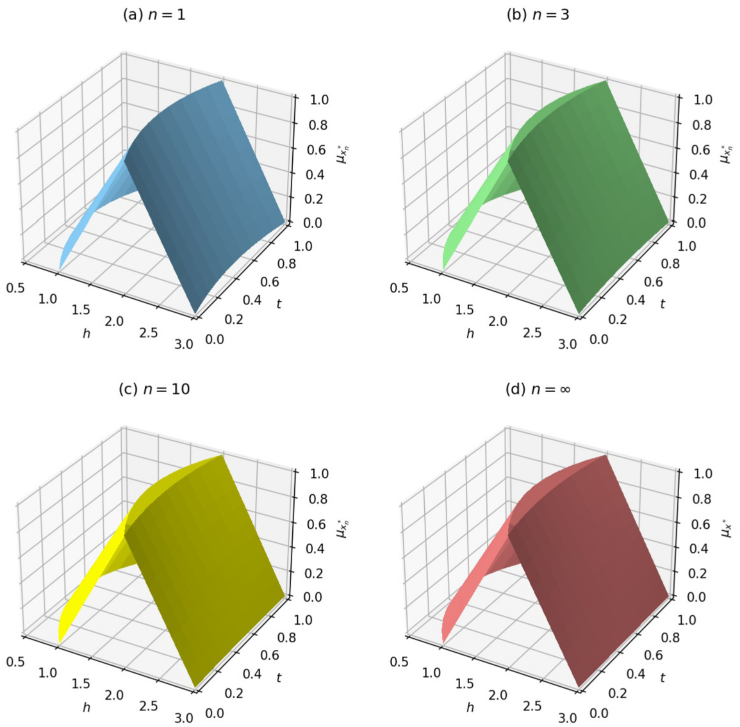

Fig. 1. The illustration for membership functions of big solutions to Example 4.1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The fact that solutions $x_n^*(t)$ are closer to the solution $x^*(t)$ as n increasing can be shown from graphs of membership functions. In this example, because perturbations are very small, graphs of different membership functions are similar. In order to more clearly demonstrate this fact, α-level sets of $x^*(t)$ and $x_n^*(t)$ are shown in Fig. 2. Fig. 2 illustrates graphs of $[x^*(t)]^\alpha = [x_1(t,\alpha), x_2(t,\alpha)]$ and $[x_n^*(t)]^\alpha = [x_1^n(t,\alpha), x_2^n(t,\alpha)]$, $(\alpha = 0,\ 0.5,\ 0.75,\ 1)$.

Fig. 2 shows that when n increases, $x_1^n(t,\alpha)$ and $x_2^n(t,\alpha)$ move closer to $x_1(t,\alpha)$ and $x_2(t,\alpha)$, respectively, for all α. It can be seen from the graph b in Fig. 2, $x_1^n(t, 0.5)$ moves closer to $x_1(t, 0.5)$ (the red line) as n increases. This indicates that the solution can keep the structure stable when there are only small perturbations on the forcing function. By Theorem 4.1, it can be concluded that $\lim_{n\to\infty} D(x_n^*(t), x^*(t)) = 0$ uniformly with respect to $t \in I$.

Example 4.2. In (1.3) and (4.2) take $I = [0, \frac{\pi}{2}]$, $f(t,\xi(t)) = -\xi(t)$, $A = B = (\alpha, 2-\alpha)$, $A_n = B_n = (\alpha + \frac{1}{n}, 2 - \alpha + \frac{1}{n})$, $\alpha \in [0,1]$. By Theorem 3.1, big solutions of (1.3) and (4.1) are

$$x_n^*(t) = w_n(t) + \int_0^{\frac{\pi}{2}} G(t,s) \otimes (-x_n^*(s))ds \ (t \in I),$$

$$x^*(t) = w(t) + \int_0^{\frac{\pi}{2}} G(t,s) \otimes (-x^*(s))ds \ (t \in I),$$

where $w_n(t) = \frac{A_n(b-t)+B_n(t-a)}{b-a}$. From Theorem 4.2, we have $\lim_{n\to\infty} D(x_n^*(t), x^*(t)) = 0$ uniformly with respect to $t \in I$.

Example 4.3. In (1.3) and (4.3), take $I = [0,1]$, $f(t,\xi(t)) = (\alpha, 2-\alpha)$, $\alpha \in [0,1]$, $f_n(t,\xi(t)) = (\alpha - \frac{1}{n}, 2 - \alpha - \frac{1}{n})$, $\alpha \in [0,1]$, $A = B = (\alpha, 2-\alpha)$, $A_n = B_n = (\alpha - \frac{1}{n}, 2-\alpha+\frac{1}{n})$, $\alpha \in [0,1]$. Then big solutions of (1.3) and (4.3) are

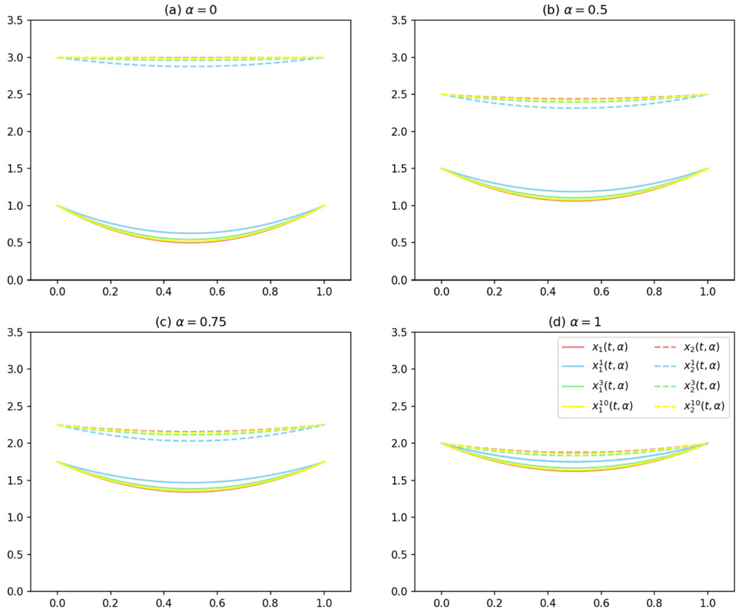

Fig. 2. The illustration for α-level sets of big solutions to Example 4.1. The order of these solid lines from top to bottom are as follows: $x_1^1(t,\alpha)$, $x_1^3(t,\alpha)$, $x_1^{10}(t,\alpha)$, $x_1(t,\alpha)$. The order of these dotted lines from top to bottom are as follows: $x_2(t,\alpha)$, $x_2^{10}(t,\alpha)$, $x_2^3(t,\alpha)$, $x_2^1(t,\alpha)$.

$x^*(t) = A + \frac{t(1-t)}{2}[(-1)\otimes f]$ and $x_n^*(t) = A_n + \frac{t(1-t)}{2}[(-1)\otimes f_n]$. It can be concluded that $\lim_{n\to\infty} D(x_n^*(t), x^*(t)) = 0$ uniformly with respect to $t \in I$.

5. The structural stability of solutions

After discussing the structural stability of big solutions, the structural stability of solutions to (1.3) is what we concerned about too. Same to [14], we will discuss the structural stability of solutions to two-point boundary value problems if given some perturbations. Similar to section 4, three corresponding cases will be discussed.

5.1. Perturbations on the forcing function

For the two-point boundary value problem (1.3), a certain perturbation to the forcing function as (4.1), we have the following theorems.

Theorem 5.1. *Suppose that f, $f_n : I\times\mathbb{R}\to \mathbf{E}_c$ satisfy conditions in Theorem 3.1 and (iv) $\lim_{n\to\infty} D(f_n(t,\xi(t)), f(t,\xi(t)) = 0$ for each $t\in I$.*

Then the solution $v_n : I\to \mathbf{D}^1$ to (4.1) and the solution $v : I\to\mathbf{D}^1$ to (1.3) satisfy $\lim_{k\to\infty} D(v_n(t), v(t)) = 0$ for each $t\in I$.

Proof. By the Theorem 3.1, (4.1) and (1.3) have solutions $v_n, v : I\to\mathbf{D}^1$ such that $[v_n(t)]^\alpha = S_\alpha^n(A,B;t)$, $[v(t)]^\alpha = S_\alpha(A,B;t)$ $(t\in I,\ 0\le\alpha\le 1)$, respectively. Denote $S_\alpha^n = S_\alpha^n(A,B;t)$, $S_\alpha = S_\alpha(A,B;t)$ for simplicity. Let $L(t,\xi(t),\alpha) = [f(t,\xi(t))]^\alpha$ and $L_n(t,\xi(t),\alpha) = [f_n(t,\xi(t))]^\alpha$ be α-level sets of f and f_n, respectively.

From (iv), $\lim_{n\to\infty} D(f_n(t,\xi(t)), f(t,\xi(t)) = \lim_{n\to\infty}\sup_{0\le\alpha\le 1} H(L_n(t,\xi(t),\alpha), L(t,\xi(t),\alpha)) = 0$ for each $t\in I$, then $\lim_{n\to\infty} H(L_n(t,\xi(t),\alpha), L(t,\xi(t),\alpha)) = 0$ uniformly with respect to $\alpha\in[0,1]$, for each $t\in I$. Therefore, for each $t\in I$, $\forall\varepsilon>0$, there exists K_1, when $n > K_1$ we have

$$L_n(t,\xi(t),\alpha)\subset L(t,\xi(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B},$$

$$L(t, \xi(t), \alpha) \subset L_n(t, \xi(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B},$$

uniformly with respect to $\alpha \in [0, 1]$, where $L(t, \xi(t), \alpha),\ L_n(t, \xi(t), \alpha) \in P_k(\mathbb{R})$, $\mathbf{B}$ is the unit ball in $\mathbb{R}$. Then we have

$$\begin{cases} \xi_n''(t) \in L_n(t, \xi_n(t), \alpha) \subset L(t, \xi_n(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}, \\ \xi_n(a) \in [A]^\alpha,\ \xi_n(b) \in [B]^\alpha, \end{cases}$$

uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

Let $\lim\limits_{n\to\infty} S_\alpha^n = \{\xi^*(t) \mid \xi^*(t) = \lim\limits_{n\to\infty} \xi_n(t),\ \xi_n(t) \in S_\alpha^n\ (t \in I)\}$. For each $\xi^*(t) \in \lim\limits_{n\to\infty} S_\alpha^n$, there exists $\xi_n(t) \in S_\alpha^n$ with $\lim\limits_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$. We need to prove $\xi^*(t) \in S_\alpha$ for each $t \in I$.

As $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$ is the α-level set of $f(t, \xi(t)) \in \mathbf{E}_c$, where $\mathbf{E}_c$ is continuous fuzzy number space, then $L(t, \xi(t), \alpha)$ is in the space $P_{kc}(\mathbb{R}) \subset P_k(\mathbb{R})$. Denote $L = L(t, \xi(t), \alpha)$ for simplicity. As $L \in P_k(\mathbb{R})$, then the graph of L: $Gr(L)$ is compact and $\inf\{||a - b|| : a \in Gr(L),\ b \in (Gr(L) + N)^c\} > \varepsilon > 0$, where $\varepsilon > 0$ is sufficiently small, $(Gr(L) + N)^c$ is the complement of $Gr(L) + N$, N is the neighborhood of θ in $\Omega \times [0,\ 1] \times \mathbb{R}$, and Ω is the open set of $\mathbb{R} \times \mathbb{R}$.

As $f(t, \xi(t))$ is continuous on $I \times \mathbb{R}$, then $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$ is upper semicontinuous uniformly with respect to $\alpha \in [0, 1]$. Therefore there exists U is the neighborhood of $(t, \xi^*(t))$ for each $t \in I$, have $L(s, \xi(s), \alpha) \subset L(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$, for $\forall(s, \xi(s)) \in U$.

For each $t \in I$, as $\lim\limits_{n\to\infty} \xi_n(t) = \xi^*(t)$, then there exists K_2, when $n > K_2$, we have $(t, \xi_n(t)) \in U$. Therefore $L(t, \xi_n(t), \alpha) \subset L(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$ when $n > K_2$.

Let $K = \max\{K_1, K_2\}$, when $n > K$, we have, for each $t \in I$, $L_n(t, \xi_n(t), \alpha) \subset L(t, \xi^*(t), \alpha) + \varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$. That is to say $(t, \xi_n(t), \alpha, \xi_n''(t)) \in Gr(L) + N$ uniformly with respect to $\alpha \in [0, 1]$. By Lemma 2.4 (Convergence Theorem), we have $\xi^{*\prime\prime}(t) \in L(t, \xi^*(t), \alpha)$ uniformly with respect to $\alpha \in [0, 1]$. As $\xi^*(t) \in \lim\limits_{n\to\infty} S_\alpha^n$, then $\xi^*(a) \in [A]^\alpha$, $\xi^*(b) \in [B]^\alpha$. So $\xi^*(t) \in S_\alpha$ uniformly with respect to $\alpha \in [0, 1]$. Then for each $t \in I$, we have $\lim\limits_{n\to\infty} S_\alpha^n \subset S_\alpha$ uniformly with respect to $\alpha \in [0, 1]$.

On the other hand, for $\forall\xi^*(t) \in S_\alpha$ for each $t \in I$, we have

$$\begin{cases} \xi^{*\prime\prime}(t) \in L(t, \xi^*(t), \alpha) \subset L_n(t, \xi^*(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}, \\ \xi^*(a) \in [A]^\alpha,\ \xi^*(b) \in [B]^\alpha, \end{cases}$$

uniformly with respect to $\alpha \in [0, 1]$, when $n > K_1$.

Let $\{\delta_n\}$ be a nonnegative sequence satisfying $\delta_n > \delta_{n+1}$ $(n = 1, 2, 3)$ and $\lim\limits_{n\to\infty} \delta_n = 0$. For $\xi^*(t)$ and each $\delta_n > 0$, there exist K_3 and a absolutely continuous function $\xi_{\delta_n}(t)$. When $n > K_3$, we have $(t, \xi_{\delta_n}(t))$ in the neighborhood U of $(t, \xi^*(t))$, have $|\xi_{\delta_n}(t) - \xi^*(t)| < \delta_n$ for each $t \in I$ and $\xi_{\delta_n}(a) \in [A]^\alpha$, $\xi_{\delta_n}(b) \in [B]^\alpha$. In other words, $(t, \xi^*(t))$ is also in the neighborhood U_{δ_n} of $(t, \xi_{\delta_n}(t))$. As $f_n(t, \xi(t))$ is continuous on $I \times \mathbb{R}$, then $L_n(t, \xi, \alpha) = [f_n(t, \xi(t))]^\alpha$ is upper semicontinuous uniformly with respect to $\alpha \in [0, 1]$. Therefore $L_n(t, \xi^*(t), \alpha) \subset L_n(t, \xi_{\delta_n}(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$. That is to say $(t, \xi^*(t), \alpha, \xi^{*\prime\prime}(t)) \in Gr(L_n(t, \xi_{\delta_n}(t), \alpha)) + N$ uniformly with respect to $\alpha \in [0, 1]$, when $n > \max\{K_1, K_3\}$. By Lemma 2.4 (Convergence Theorem), we have $\xi_{\delta_n}''(t) \in L_n(t, \xi_{\delta_n}(t), \alpha)$ uniformly with respect to $\alpha \in [0, 1]$. So $\xi_{\delta_n}(t) \in S_\alpha^n$ uniformly with respect to $\alpha \in [0, 1]$. Then we have $S_\alpha \subset \lim\limits_{n\to\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0, 1]$. Therefore, for each $t \in I$, $S_\alpha = \lim\limits_{n\to\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0, 1]$.

Then $\lim\limits_{n\to\infty} D(v_n, v) = \lim\limits_{n\to\infty} \sup\limits_{0\le\alpha\le1} H(S_\alpha^n, S_\alpha) = 0$ for each $t \in I$. □

Theorem 5.2. *Suppose that $f,\ f_n : I \times \mathbb{R} \to \mathbf{E}_c$ satisfy conditions in Theorem 3.1 and (iv) $\{f_n(t, \xi(t))\}$ is monotone with respect to n and $\lim\limits_{n\to\infty} D(f_n(t, \xi(t)), f(t, \xi(t)) = 0$ for each $t \in I$.*

Then the solution $v_n : I \to \mathbf{D}^1$ to (4.1) and the solution $v : I \to \mathbf{D}^1$ to (1.3) satisfy $\lim\limits_{n\to\infty} D(v_n(t), v(t)) = 0$ uniformly with respect to $t \in I$.

Proof. By the Theorem 3.1, (4.1) and (1.3) have solutions $v_n, v : I \to \mathbf{D}^1$ such that $[v_n(t)]^\alpha = S_\alpha^n,\ [v(t)]^\alpha = S_\alpha$ $(t \in I,\ 0 \le \alpha \le 1)$, respectively. Let $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$ and $L_n(t, \xi(t), \alpha) = [f_n(t, \xi(t))]^\alpha$ be α-level sets of f and f_n, respectively.

As $\{f_n(t,\xi(t))\}$ is monotone, two situations will be discussed to prove that $\lim_{n\to\infty} D(v_n(t), v(t)) = 0$ uniformly with respect to $t \in I$.

(1), If $\{f_n(t,\xi(t))\}$ is monotone decreasing, i.e., $f_{n+1} \subset f_n$, $(n = 1, 2, ...)$.

From condition (iv), we have

$$L(t,\xi(t),\alpha) \subset \cdots \subset L_{n+1}(t,\xi(t),\alpha) \subset L_n(t,\xi(t),\alpha) \subset \cdots \subset L_1(t,\xi(t),\alpha),$$

uniformly with respect to $\alpha \in [0, 1]$, where $L(t,\xi(t),\alpha),\ L_n(t,\xi(t),\alpha) \in P_k(\mathbb{R})$.

For each $t \in I$, $\forall \xi^*(t) \in S_\alpha$, we have

$$\begin{cases} \xi^{*\prime\prime}(t) \in L(t,\xi^*(t),\alpha) \subset \cdots \subset L_n(t,\xi^*(t),\alpha) \subset \cdots \subset L_1(t,\xi^*(t),\alpha), \\ \xi^*(a) \in [A]^\alpha,\ \xi^*(b) \in [B]^\alpha, \end{cases}$$

then $\xi^*(t) \in S_\alpha^n$ and $S_\alpha \subset \cdots \subset S_\alpha^n \subset \cdots \subset S_\alpha^1$. It can be concluded that $S_\alpha \subset \bigcap_{n=1}^{\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0, 1]$.

For $\forall \xi^*(t) \in \bigcap_{n=1}^{\infty} S_\alpha^n$, there exists $\xi_n(t) \in S_\alpha^n$ with $\lim_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$.

As $\lim_{n\to\infty} D(f_n(t,\xi(t)), f(t,\xi(t))) = 0$ for each $t \in I$, then

$$H(L_n(t,\xi(t),\alpha), L(t,\xi(t),\alpha)) \to 0,$$

uniformly with respect to $\alpha \in [0, 1]$. Further $L_n(t,\xi(t),\alpha) \subset L(t,\xi(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$ when n is sufficiently large.

As $f(t,\xi(t))$ is continuous on $I \times \mathbb{R}$, then $L(t,\xi(t),\alpha) = [f(t,\xi(t))]^\alpha$ is upper semicontinuous uniformly with respect to $\alpha \in [0, 1]$. Same to the proof of Theorem 5.1, there exists U is the neighborhood of $(t,\xi^*(t))$, have $L(s,\xi(s),\alpha) \subset L(t,\xi^*(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$ for $\forall (s,\xi(s)) \in U$, $\varepsilon > 0$ is sufficiently small.

As $\lim_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$, then $(t,\xi_n(t)) \in U$ when n is sufficiently large. Therefore $L(t,\xi_n(t),\alpha) \subset L(t,\xi^*(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$.

Furthermore, for each $t \in I$, $L_n(t,\xi_n(t),\alpha) \subset L(t,\xi^*(t),\alpha) + \varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$. That is to say $(t,\xi_n(t),\alpha,\xi_n''(t)) \in Gr(L) + N$ uniformly with respect to $\alpha \in [0, 1]$, where N is defined in the proof of Theorem 5.1. By Lemma 2.4 (Convergence Theorem), we have $\xi^{*\prime\prime}(t) \in L(t,\xi^*(t),\alpha)$ uniformly with respect to $\alpha \in [0, 1]$.

As $\xi^*(t) \in \bigcap_{n=1}^{\infty} S_\alpha^n$, then $\xi^*(a) \in [A]^\alpha$, $\xi^*(b) \in [B]^\alpha$. So $\xi^*(t) \in S_\alpha$ uniformly with respect to $\alpha \in [0, 1]$.

Then we have $S_\alpha = \bigcap_{n=1}^{\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0, 1]$ for each $t \in I$.

Let $\varphi_n(t,x,\alpha) = \sigma_{S_\alpha^n}(x)$, $\varphi(t,x,\alpha) = \sigma_{S_\alpha}(x)$, $(t,x,\alpha) \in I \times \{\pm 1\} \times [0, 1]$, where $\sigma_A(x) = \sup\{\langle x, y\rangle :\ y \in A\}$ be a support function and $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}$, then $\varphi_n(t,x,\alpha)$, $\varphi(t,x,\alpha)$ are continuous with respect to (t,x), and $\lim_{n\to\infty} \varphi_n(t,x,\alpha) = \varphi(t,x,\alpha)$ uniformly with respect to $\alpha \in [0, 1]$ for each (t,x). For each fixed (t,x), $\{\varphi_n(t,x,\alpha)\}$ is monotonously decreasing, then by Dini Theorem, $\lim_{n\to\infty} \varphi_n(t,x,\alpha) = \varphi(t,x,\alpha)$ uniformly with respect to $(t,x,\alpha) \in I \times \{\pm 1\} \times [0, 1]$.

Therefore,

$$H(S_\alpha^n, S_\alpha) = \sup_{x\in\{\pm 1\}} \{|\sigma_{S_\alpha^n}(x) - \sigma_{S_\alpha}(x)| :\ x = \pm 1\} \to 0,\ n \to \infty,$$

uniformly with respect to $(t,\alpha) \in I \times [0, 1]$.

Then $\lim_{n\to\infty} D(v_n, v) = \lim_{n\to\infty} \sup_{0\le\alpha\le 1} H(S_\alpha^n, S_\alpha) = 0$ uniformly with respect to $t \in I$.

(2), If $\{f_n(t,\xi(t))\}$ is monotone increasing, i.e., $f_n \subset f_{n+1}$, $(n = 1, 2, ...)$.

Similar to the discussion of (1), we can get the corresponding conclusion.

Therefore, $\lim_{n\to\infty} D(v_n(t), v(t)) = \lim_{n\to\infty} \sup_{0\le\alpha\le 1} H(S_\alpha^n, S_\alpha) = 0$ uniformly with respect to $t \in I$ when $\{f_n(t,\xi(t))\}$ is monotone. □

5.2. Perturbations on boundary conditions

For the two-point boundary value problem (1.3), if perturbations on boundary conditions as (4.2), then we have the following theorem.

Theorem 5.3. *Suppose that $f : I \times \mathbb{R} \to \mathbf{E}_c$ and boundary conditions satisfy conditions in Theorem 4.2.*

Then the solution $v_n : I \to \mathbf{D}^1$ to (4.2) and the solution $v : I \to \mathbf{D}^1$ to (1.3) satisfy $\lim_{k\to\infty} D(v_n(t), v(t)) = 0$ for each $t \in I$.

Proof. By the Theorem 3.1, (4.2) and (1.3) have solutions $v_n, v : I \to \mathbf{D}^1$ such that $[v_n(t)]^\alpha = S^n_\alpha(A_n, B_n; t)$, $[v(t)]^\alpha = S_\alpha(A, B; t)$ $(t \in I,\ 0 \le \alpha \le 1)$, respectively. Denote $S^n_\alpha = S^n_\alpha(A_n, B_n; t)$, $S_\alpha = S_\alpha(A, B; t)$ for simplicity. Let $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha = (f_1(\alpha), f_2(\alpha))$ be the α-level set of $f(t, \xi(t))$.

For $\forall \xi^*(t) \in \lim_{n\to\infty} S^n_\alpha$, there exists $\xi_n(t) \in S^n_\alpha$ with $\lim_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$. We have

$$\begin{cases} \xi_n{}''(t) \in L(t, \xi_n(t), \alpha), \\ \xi_n(a) \in [A_n]^\alpha,\ \xi_n(b) \in [B_n]^\alpha. \end{cases}$$

As $f(t, \xi(t))$ is continuous on $I \times \mathbb{R}$, then $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$ is upper semicontinuous uniformly with respect to $\alpha \in [0, 1]$. Same to the proof of Theorem 5.1, there exists U is the neighborhood of $(t, \xi^*(t))$, have $L(s, \xi(s), \alpha) \subset L(t, \xi^*(t), \alpha) + \varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$, for $\forall (s, \xi(s)) \in U$, $\varepsilon > 0$ is sufficiently small.

As $\lim_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$, then $(t, \xi_n(t)) \in U$ when n is sufficiently large. Therefore $L(t, \xi_n(t), \alpha) \subset L(t, \xi^*(t), \alpha) + \varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$. That is to say $(t, \xi_n(t), \alpha, \xi_n{}''(t)) \in Gr(L) + N$ uniformly with respect to $\alpha \in [0, 1]$, where N is defined in the proof of Theorem 5.1. By Lemma 2.4 (Convergence Theorem), we have $\xi^{*\prime\prime}(t) \in L(t, \xi^*(t), \alpha)$ uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

As $\lim_{n\to\infty} D(A_n, A) = 0$, $\lim_{n\to\infty} D(B_n, B) = 0$ and $\xi^*(t) \in \lim_{n\to\infty} S^n_\alpha$, then $\xi^*(a) = \lim_{n\to\infty} \xi_n(a) \in \lim_{n\to\infty} [A_n]^\alpha = [A]^\alpha$ uniformly with respect to $\alpha \in [0, 1]$, $\xi^*(b) = \lim_{n\to\infty} \xi_n(b) \in \lim_{n\to\infty} [B_n]^\alpha = [B]^\alpha$ uniformly with respect to $\alpha \in [0, 1]$. So $\xi^*(t) \in S_\alpha$ uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

Then we have $\lim_{n\to\infty} S^n_\alpha \subset S_\alpha$ uniformly with respect to $\alpha \in [0, 1]$.

On the other hand, for each $t \in I$, $\forall \xi^*(t) \in S_\alpha$, we have

$$\begin{cases} \xi^{*\prime\prime}(t) \in L(t, \xi^*(t), \alpha), \\ \xi^*(a) \in [A]^\alpha,\ \xi^*(b) \in [B]^\alpha, \end{cases}$$

uniformly with respect to $\alpha \in [0, 1]$.

Same to the proof of Theorem 5.1, for each $t \in I$, $\delta_n > 0$ and $\xi^*(t)$, there exists $(t, \xi_{\delta_n}(t), \alpha)$ in the neighborhood U of $(t, \xi^*(t))$, have $|\xi_{\delta_n}(t) - \xi^*(t)| < \delta_n$ for each $t \in I$ and $\xi_{\delta_n}(a) \in [A_n]^\alpha$, $\xi_{\delta_n}(b) \in [B_n]^\alpha$. Then $(t, \xi^*(t))$ is also in the neighborhood U_{δ_n} of $(t, \xi_{\delta_n}(t))$. As $f(t, \xi(t))$ is continuous on $I \times \mathbb{R}$, then $L(t, \xi(t), \alpha) = [f(t, \xi(t))]^\alpha$ is upper semicontinuous uniformly with respect to $\alpha \in [0, 1]$. Therefore $L(t, \xi^*(t), \alpha) \subset L(t, \xi_{\delta_n}(t), \alpha) + \frac{1}{2}\varepsilon\mathbf{B}$ uniformly with respect to $\alpha \in [0, 1]$. That is to say $(t, \xi^*(t), \alpha, \xi^{*\prime}(t)) \in Gr(L(t, \xi_{\delta_n}(t), \alpha)) + N$ uniformly with respect to $\alpha \in [0, 1]$. By Lemma 2.4 (Convergence Theorem), we have $\xi_{\delta_n}{}''(t) \in L(t, \xi_{\delta_n}(t), \alpha)$ uniformly with respect to $\alpha \in [0, 1]$. So $\xi_{\delta_n}(t) \in S^n_\alpha$ uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

Then we have $S_\alpha \subset \lim_{n\to\infty} S^n_\alpha$ uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

Therefore, $S_\alpha = \lim_{n\to\infty} S^n_\alpha$ uniformly with respect to $\alpha \in [0, 1]$, for each $t \in I$.

Then $\lim_{n\to\infty} D(v_n, v) = \lim_{n\to\infty} \sup_{0\le\alpha\le1} H(S^n_\alpha, S_\alpha) = 0$ for each $t \in I$. □

5.3. Perturbations on the forcing function and boundary conditions

If perturbations occur both on boundary conditions and the forcing function as (4.3) for the two-point boundary value problem (1.3), then we have the following theorem.

Theorem 5.4. *Suppose that $f,\ f_n: I \times \mathbb{R} \to \mathbf{E}_c$ and boundary conditions satisfy conditions in Theorem 3.1, and*

(iv) $\lim\limits_{n\to\infty} D(f_n(t,\xi(t)), f(t,\xi(t)) = 0$ *for each* $t \in I$,

(v) $\lim\limits_{n\to\infty} D(A_n, A) = 0,\ \lim\limits_{n\to\infty} D(B_n, B) = 0.$

Then the solution $v_n : I \to \mathbf{D}^1$ *to (4.3) and the solution* $v : I \to \mathbf{D}^1$ *to (1.3) satisfy* $\lim\limits_{k\to\infty} D(v_n(t), v(t)) = 0$ *for each* $t \in I$.

Proof. By the Theorem 3.1, (4.3) and (1.3) have solutions $v_n, v : I \to \mathbf{D}^1$ such that $[v_n(t)]^\alpha = S_\alpha^n,\ [v(t)]^\alpha = S_\alpha$ $(t \in I,\ 0 \le \alpha \le 1)$, respectively. Let $L(t,\xi(t),\alpha) = [f(t,\xi(t))]^\alpha$ and $L_n(t,\xi(t),\alpha) = [f_n(t,\xi(t))]^\alpha$ be α-level sets of f and f_n, respectively.

Similar to the proof of Theorem 5.1, we have

$$\begin{cases} \xi_n{}''(t) \in L_n(t,\xi_n(t),\alpha) \subset L(t,\xi_n(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B}, \\ \xi_n(a) \in [A_n]^\alpha,\ \xi_n(b) \in [B_n]^\alpha, \end{cases}$$

uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

Let $\lim\limits_{n\to\infty} S_\alpha^n = \{\xi^*(t) \mid \xi^*(t) = \lim\limits_{n\to\infty} \xi_n(t),\ \xi_n(t) \in S_\alpha^n\ (t \in I)\}$. For each $\xi^*(t) \in \lim\limits_{n\to\infty} S_\alpha^n$, there exists $\xi_n(t) \in S_\alpha^n$ with $\lim\limits_{n\to\infty} \xi_n(t) = \xi^*(t)$ for each $t \in I$. We need to prove $\xi^*(t) \in S_\alpha$ for each $t \in I$.

Same to the proof of Theorem 5.1, we have $\xi^{*\prime\prime}(t) \in L(t,\xi^*(t),\alpha)$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$. Same to the proof of Theorem 5.3, we have $\xi^*(a) = \lim\limits_{n\to\infty} \xi_n(a) \in \lim\limits_{n\to\infty} [A_n]^\alpha = [A]^\alpha$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$, $\xi^*(b) = \lim\limits_{n\to\infty} \xi_n(b) \in \lim\limits_{n\to\infty} [B_n]^\alpha = [B]^\alpha$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$. So $\xi^*(t) \in S_\alpha$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

Then we have $\lim\limits_{n\to\infty} S_\alpha^n \subset S_\alpha$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

On the other hand, for each $t \in I$, $\forall \xi^*(t) \in S_\alpha$, we have

$$\begin{cases} \xi^{*\prime\prime}(t) \in L(t,\xi^*(t),\alpha) \subset L_n(t,\xi^*(t),\alpha) + \frac{1}{2}\varepsilon\mathbf{B}, \\ \xi^*(a) \in [A]^\alpha,\ \xi^*(b) \in [B]^\alpha, \end{cases}$$

uniformly with respect to $\alpha \in [0,1]$.

Same to the proof of Theorem 5.1, for each $t \in I$, $\delta_n > 0$ and $\xi^*(t)$, there exists $(t, \xi_{\delta_n}(t))$ in the neighborhood U of $(t,\xi^*(t))$, have $|\xi_{\delta_n}(t) - \xi^*(t)| < \delta_n$ for each $t \in I$ and $\xi_{\delta_n}(a) \in [A_n]^\alpha$ uniformly with respect to $\alpha \in [0,1]$, $\xi_{\delta_n}(b) \in [B_n]^\alpha$ uniformly with respect to $\alpha \in [0,1]$. Same to the proof of Theorem 5.3, we have $\xi_{\delta_n}(t) \in S_\alpha^n$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

Then we have $S_\alpha \subset \lim\limits_{n\to\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

Therefore, $S_\alpha = \lim\limits_{n\to\infty} S_\alpha^n$ uniformly with respect to $\alpha \in [0,1]$, for each $t \in I$.

Then $\lim\limits_{n\to\infty} D(v_n, v) = \lim\limits_{n\to\infty} \sup\limits_{0\le\alpha\le1} H(S_\alpha^n, S_\alpha) = 0$ for each $t \in I$. □

Remark 5.1. If $A_n,\ B_n$ is monotone and $\lim\limits_{n\to\infty} A_n = A,\ \lim\limits_{n\to\infty} B_n = B$. Then the conclusions of Theorem 5.3 and Theorem 5.4 are also valid.

Remark 5.2. If the monotonicity of $A_n,\ B_n$ and f_n are consistent, then solutions converge uniformly with respect to $t \in I$. The proof of this claim can be derived from Theorem 5.2-5.4.

Remark 5.3. The solution $v(t)$ for fuzzy differential inclusion problems (1.3) could not directly be represented by the integral of $f(t,\xi(t))$ like the big solution $x(t)$. So the way to discuss of structural stability of the solution in section 5 is different from the method to prove the structural stability of big solutions in section 4.

Example 5.1. In (1.3) and (4.1), take $I = [0,1],\ A = B \in \mathbf{E}_c,\ f(t,\xi(t)) = (\alpha, 2-\alpha),\ \alpha \in [0,1],\ f_n(t,\xi(t)) = (\alpha - \frac{1}{n}, 2-\alpha-\frac{1}{n}),\ \alpha \in [0,1]$. Then solutions of (1.3) and (4.1) are $v(t) = A + \frac{t(1-t)}{2}[(-1)\otimes f],\ v_n(t) = A + \frac{t(1-t)}{2}[(-1)\otimes f_n]$. It can be concluded that $\lim\limits_{n\to\infty} D(v_n(t), v(t)) = 0$ uniformly with respect to $t \in I$.

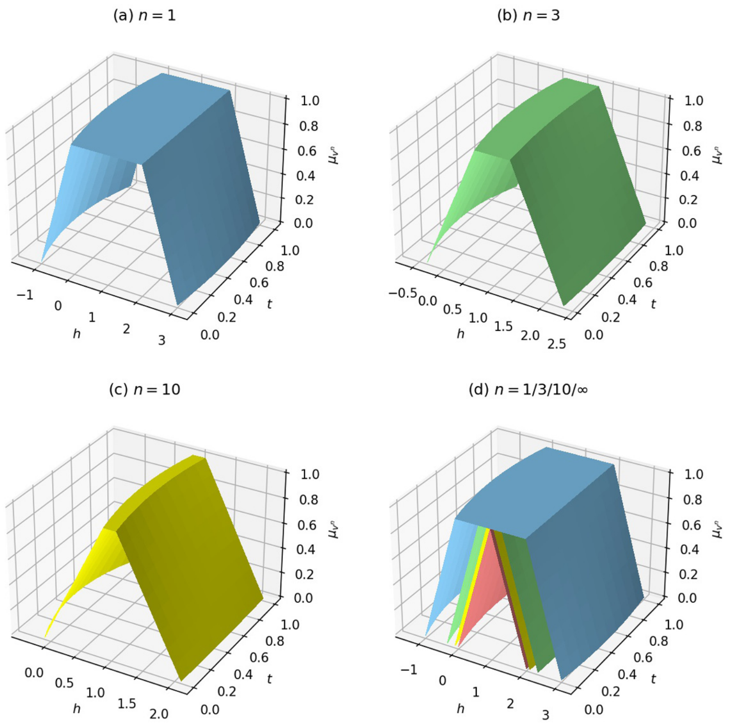

Fig. 3. The illustration for membership functions of solutions to Example 5.3.

Example 5.2. In (1.3) and (4.2) take $I = [0, \frac{\pi}{2}]$, $f(t, \xi(t)) = -\xi(t)$, $A = B = (\alpha, 2 - \alpha)$, $A_n = B_n = (\alpha - \frac{1}{n}, 2 - \alpha - \frac{1}{n})$, $\alpha \in [0, 1]$. Then solutions of (1.3) and (4.1) are $v(t) = A(\cos t + \sin t)$, $v_n(t) = A_n(\cos t + \sin t)$. It can be concluded that $\lim_{n\to\infty} D(v_n(t), v(t)) = 0$ uniformly with respect to $t \in I$.

Example 5.3. In (1.3) and (4.1), take $I = [0, 1]$, $f(t, \xi(t)) = (\alpha, 2 - \alpha)$, $\alpha \in [0, 1]$, $f_n(t, \xi(t)) = (\alpha - \frac{1}{n}, 2 - \alpha - \frac{1}{n})$, $\alpha \in [0, 1]$, $A = B = (\alpha, 2 - \alpha)$, $A_n = B_n = (\alpha - \frac{1}{n}, 2 - \alpha + \frac{1}{n})$, $\alpha \in [0, 1]$. In this example the big solutions are equal to its corresponding solution. Solutions of (1.3) and (4.1) are $v(t) = x^*(t) = A + \frac{t(1-t)}{2}[(-1) \otimes f]$ and $v_n(t) = x_n^*(t) = A_n + \frac{t(1-t)}{2}[(-1) \otimes f_n]$.

Membership functions $\mu_v(h, t)$ and $\mu_{v_n}(h, t)$ $(n = 1, 3, 10)$ of solutions $v(t)$ and $v_n(t)$ $(n = 1, 3, 10)$ are shown in Fig. 3. The graph d in Fig. 3 also shows that $\mu_{v_n}(h, t)$ move closer to $\mu_v(h, t)$ (the red curved surface) as n increasing.

The fact that solutions $v_n(t)$ are closer to the solution $v(t)$ as n increasing can also be shown from graphs of α-level sets of $v(t)$ and $v_n(t)$. Fig. 4 illustrates graphs of $[v(t)]^\alpha = [v_1(t, \alpha), v_2(t, \alpha)]$ and $[v_n(t)]^\alpha = [v_1^n(t, \alpha), v_2^n(t, \alpha)]$, $(\alpha = 0,\ 0.5,\ 0.75,\ 1)$.

Fig. 4 shows that when n increases, $v_1^n(t, \alpha)$ and $v_2^n(t, \alpha)$ move closer to $v_1(t, \alpha)$ and $v_2(t, \alpha)$, respectively, for all α. It can be seen from the graph c in Fig. 2, $v_2^n(t, 0.75)$ moves closer to $v_2(t, 0.75)$ (the red dotted line) as n increases. This indicates that the solution can keep the structure stable when there are only small perturbations on the forcing function and boundary conditions. By Theorem 5.4, it can be concluded that $\lim_{n\to\infty} D(v_n(t), v(t)) = 0$ uniformly with respect to $t \in I$.

6. Conclusion and future expectations

This paper investigated the structural stability for two-point boundary value problem of FDEs understood as corresponding differential inclusions. In the sense of differential inclusion, the structural stability of the big solution and the solution have been established respectively. The existence of the big solution is essential in the proof of the existence to the solution. On the other hand, the big solution to the two-point boundary value problem actually the solution to the

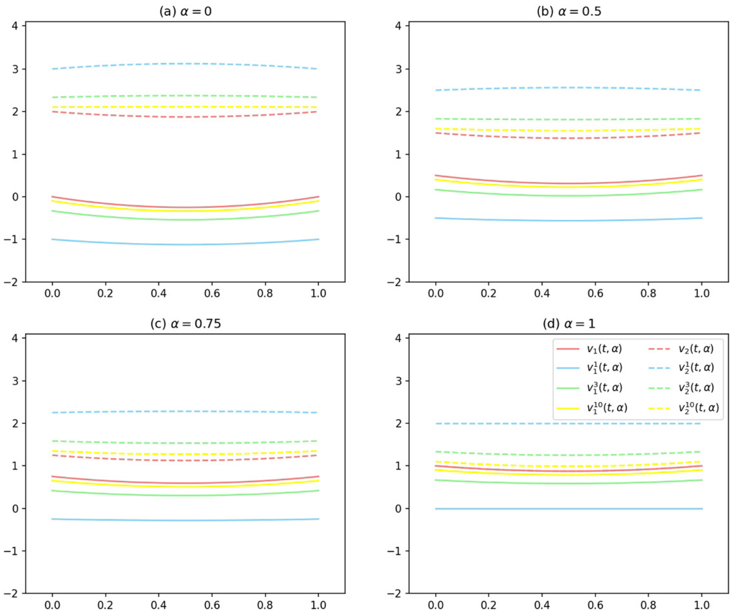

Fig. 4. The illustration for α-level sets of solutions to Example 5.3. The order of these solid lines from top to bottom are as follows: $v_1(t,\alpha)$, $v_1^{10}(t,\alpha)$, $v_1^3(t,\alpha)$, $v_1^1(t,\alpha)$. The order of these dotted lines from top to bottom are as follows: $v_2^1(t,\alpha)$, $v_2^3(t,\alpha)$, $v_2^{10}(t,\alpha)$, $v_2(t,\alpha)$.

corresponding fuzzy integral equation by extending the forcing function. To discuss the structural stability of the big solution is to discuss the structural stability of the fuzzy integral equation. The Convergence Theorem in differential inclusion theory is crucial to prove the structural stability of the solution. The monotonicity of forcing functions and boundary conditions is close to the uniformity of convergence to solutions and big solutions by Dini Theorem.

In this paper, the undamped situation to two-point boundary value problem of second-order FDEs has been considered. For the damped situation, different strategies are used to prove the existence of solution. Therefore, methods and techniques of this paper can not be directly applied to the damped situation. In the future work, the general situation of these problems is also interesting to study.

Declaration of competing interest

This study is no conflict of interest.

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